Notes on Supersymmetry
$N = 1$ SUSY Effective Actions

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Contents

1. Introduction

In preceding note we learned that the most general renormalizable field theory equiped with $N = 1$ susy can be expressed fully in terms of chiral superfield Φ subjected to the constraint $\overline{\mathrm{D}}_{\dot\alpha}\Phi = 0$, its conjugation $\Phi^\dagger$, and the vector superfield V satisfying $V^\dagger = V$. The Lagrangian of this theory can be written as

$$\begin{aligned}\mathscr{L} = 2\mathrm{Re}\bigg[\frac{\tau}{16\pi i k}\int \mathrm{d}^2\theta\ \mathrm{tr}\, W^\alpha W_\alpha\bigg] + \int \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, \Phi^\dagger e^{2V}\Phi \\ + 2\mathrm{Re}\bigg[\int \mathrm{d}^2\theta\,\Big(\lambda\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3\Big)\bigg],\end{aligned} \tag{1}$$

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where

$$\tau = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi \mathrm{i}}{g^2} \tag{2}$$

is the generalized gauge coupling, k is the normalization factor from $\mathrm{tr}\,(T^a T^b) = k\delta^{ab}$ with T^a's being generators of gauge group.

However, we are also interested in nonrenormalizable theories. For instance, when we are concerned with low energy dynamics of the theory, the most powerful approach is to write down the most general Lagrangian compatible with the symmetry of the theory, including both renormalizable and nonrenormalizable terms. Such Lagrangian can be obtained from a fundamental renormalizable theory by integrating out high energy modes. However, this is usually very hard to do in practice, unless the theory is weakly coupled along the whole energy scale under consideration. In particular, if the low energy dynamics is strongly coupled and nonperturbative from the viewpoint of the original theory, the effective action method will be almost unique, as an analytic approach.

2. Effective Action in Perturbation Theory

Now we are going to study the effective action of $N = 1$ theories from the viewpoint of holomorphy. It turns out that the celebrated non-renormalization theorem can be derived intuitively in the context of effective actions without invoking supergraph techniques.

It should be clarified at the beginning that the effective action we are going to study in this section is in the sense of Wilson's approach to renormalization group. That is, the effective action, which depends on a characteristic energy scale E, is obtained from the underlying ultra-violet theory defined at a cut-off scale Λ_c, by integrating out the modes with an energy shell between E and Λ_c. This is conventionally called the Wilsonian effective action in literature, and is in general different from another "effective action", the generating functional of all 1-particle-irreducible (1PI) Green's functions, which will be referred to as 1PI effective action. By definition, the 1PI effective action is obtained by integrating out all modes below the cut-off scale down to zero energy. As was pointed out in [3], the distinction between these two types of effective actions is crucial when the theory contains interacting massless modes, in which case the 1PI effective action would suffer from IR ambiguities and lead to the so-called holomorphic anomalies. On the contrary, since the Wilsonian effective action involves the momentum within an energy shell bounded from above and below, so it does not have IR problem even when there are massless modes in the theory.

2.1 Non-renormalization theorem

Let's consider a general $N = 1$ theory including chiral superfields and vector superfields. Since we are doing perturbation theory, the θ_{YM}-angle can be dropped. We assign standard kinetic terms for chiral supefield Φ and vector superfield V, and the remaining terms can be generally parameterized by two functions $\mathscr{A}[\Phi,\Phi^\dagger,V]$ and $\mathscr{B}(\Phi,W_\alpha)$, as follows,

$$\boxed{\begin{aligned}\mathscr{L}=&\frac{\tau}{8\pi\mathrm{i}k}\,\mathrm{Re}\int\mathrm{d}^2\theta\ \mathrm{tr}\,W^\alpha W_\alpha+\int\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\Phi^\dagger e^{2V}\Phi\\&+\int\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\mathscr{A}[\Phi,\Phi^\dagger,V]+2\mathrm{Re}\int\mathrm{d}^2\theta\,\mathscr{B}(\Phi,W_\alpha).\end{aligned}}\tag{3}$$

Here we use square bracket for $\mathscr{A}[\Phi,\Phi^\dagger,V]$ to show that $\mathscr{A}$ is a function of indicated superfiels and their (super)derivatives, while $\mathscr{B}(\Phi,W_\alpha)$ can only depend on Φ and W_α, but not on their conjugates or derivatives. The superpotential then is given by $\mathcal{W}(\Phi)=\mathscr{B}(\Phi,0)$.

Now we are going to study the scale dependence of this theory. Then, the Lagrangian above should be interpreted as the definition of the theory at a UV scale Λ_c which we will refer to as cut-off scale. What we want is the behavior of this theory at the energy scale E below the cut-off scale Λ_c. To achieve this, the standard procedure of Wilson's approach is to integrate out all modes between E and Λ_c. The resulted Lagrangian is generally different from the original one (3). A remarkable result we will prove in the following is that, to any finite order in perturbation theory, the Wilsonian effective Lagrangian at energy scale E is given by,

$$\begin{aligned}\mathscr{L}_E=&\frac{\tau_E}{8\pi\mathrm{i}k}\,\mathrm{Re}\int\mathrm{d}^2\theta\ \mathrm{tr}\,W^\alpha W_\alpha+\int\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\Phi^\dagger e^{2V}\Phi\\&+\int\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\mathscr{A}_E[\Phi,\Phi^\dagger,V]+2\mathrm{Re}\int\mathrm{d}^2\theta\,\mathscr{B}(\Phi,W_\alpha).\end{aligned}\tag{4}$$

That is, the kinetic term of vector superfield receive 1-loop corrections only, with the gauge coupling τ replaced by the corresponding 1-loop running coupling τ_E. We know from ordinary field theory that the gauge coupling β function of a non-Abelian theory with chiral (or Majorana) fermions in representation r_f and complex scalars in representation r_s is given by,

$$\beta(g)=-\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(\mathrm{Ad})-\frac{2}{3}C(r_f)-\frac{1}{3}C(r_s)\right],\tag{5}$$

where $C_2(\mathrm{Ad})$ and $C(r)$ are defined through $f^{acd}f^{bcd}=C_2(\mathrm{Ad})\delta^{ab}$ and $\mathrm{tr}\,(T_r^aT_r^b)=C(r)\delta^{ab}$ with T_r^a the matrix of representation r of corresponding gauge generator. In our current theory, let the chiral superfield Φ be in representation r, then we have $r_s=r$ and $r_f=r\oplus\mathrm{Ad}$ where Ad denotes adjoint representation. Note further that $C_2(\mathrm{Ad})=C(\mathrm{Ad})$, so we have $\beta(g)=-\frac{1}{(4\pi)^2}g^3\big[3C(\mathrm{Ad})-C(r)\big]$. Then it can be easily solved that $\frac{1}{g_E^2}=\frac{1}{g^2}+\frac{1}{8\pi^2}[3C_2(\mathrm{Ad})-C(r)]\log(E/\Lambda_c)$. Meanwhile, there is no 1-loop correction to the topological angle θ_{YM}. Thus,

$$\boxed{\tau_E=\tau+\frac{\mathrm{i}\big[3C(\mathrm{Ad})-C(r)\big]}{2\pi}\log\frac{E}{\Lambda_c}.}\tag{6}$$

Furthermore, the $\mathscr{B}$-function in original Lagrangian (3) does not receive any quantum corrections in perturbation theory. In particular, this implies that the superpotential $\mathcal{W}(\Phi) = \mathscr{B}(\Phi, 0)$ is not renormalized. We have proved this result for renormalizable Lagrangian with supergraph technique, which was originally done in [4]. In what follows, we will derive this more general result in the context of Wilsonian effective action following the holomorphy approach of Seiberg [6] and its generalized version [8].

Before getting into the proof, one should be aware that this non-renormalization for superpotential is subject to the assumption of perturbativity, namely the weak couplings in the whole range of energies from E to Λ_c. When there is no non-Abelian gauge fields, this must be true if the theory is perturbative at Λ_c, since we know all renormalizable theories without non-Abelian gauge fields are IR free, while nonrenormalizable couplings are irrelevant at IR. So the scale E of effective action can be put arbitrarily low. This is also the case even when non-Abelian gauge fields are prensent, but with enough number of matter fields to alter the sign of beta function, or, with a Higgs mechanism to break the non-Abelian gauge symmetry to an IR free theory, as the case of electroweak theory. However, when the non-Abelian gauge symmetry is unbroken, the theory would become strongly coupled at and below an energy Λ_g. In this case the effective action can be applied only when $\Lambda_g \ll E < \Lambda_c$.

The key observation of Seiberg's proof is that one can treat couplings of cut-off theory in (3) as background values of some fields. A prototype of such treatment is the string coupling in string theory, which can actually be identified as the vacuum expectation value of dilaton field. The point of this treatment in current theory is that, since we are only concerned with physics below Λ_c, we are free to design a new theory at energies far above Λ_c as long as it can reproduce (3) at Λ_c. In this way, the new theory we designed should lead to the same RG behavior as (3). Since the perturbation theory already holds at and below Λ_c, the RG flow should not have exotic behavior, thus the argument above is justified. With this in mind, we promote the gauge coupling τ to a chiral superfield T, various coefficients g_i in $\mathscr{A}$ to vector superfields G_i, and various coefficients λ_i in $\mathscr{B}$ to chiral superfields L_i. Then, the Lagrangian (3) of cut-off theory can be rewritten as,

$$\begin{aligned}\mathscr{L} = \frac{1}{8\pi \mathrm{i} k} \mathrm{Re} \int \mathrm{d}^2\theta\, T\, \mathrm{tr}\, W^\alpha W_\alpha + \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \Phi^\dagger e^{2V} \Phi \\ + \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \mathscr{A}_E[\Phi, \Phi^\dagger, V; G_i] + 2\mathrm{Re} \int \mathrm{d}^2\theta\, \mathscr{B}(\Phi, W_\alpha; L_i).\end{aligned} \tag{7}$$

Clearly we can design the theory such that T and L_i are frozen at cut-off scale Λ_c with values $T = \tau$ and $L_i = \lambda_i$, then the original theory (3) is naturally recovered.

Now, after integrating out modes from cut-off scale Λ_c down to E, the resulted effective

Lagrangian $\mathscr{L}_E$ can be generally written as,

$$\begin{aligned}\mathscr{L}_E = \frac{1}{8\pi \mathrm{i} k}\,\mathrm{Re}\int \mathrm{d}^2\theta\, T\,\mathrm{tr}\, W^\alpha W_\alpha + \int \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, \Phi^\dagger e^{2V}\Phi \\ + \int \mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, \mathscr{A}_E[\Phi,\Phi^\dagger,V;T,T^\dagger,L_i,L_i^\dagger,G_i] \\ + 2\mathrm{Re}\int \mathrm{d}^2\theta\, \mathscr{B}_E(\Phi,W_\alpha,T;L_i,G_i).\end{aligned} \tag{8}$$

To constrain the effective lagrangian (8), we study various symmetries of the cut-off theory. Firstly, it appears odd that $\mathscr{B}_E$ can depend on vector superfields G_i. We spell out this dependence explicitly because $\mathscr{B}_E$ may depend on some chiral superfields generated from L_i. In fact, vector superfields G_i are always associated with a generalized gauge transformation, $G_i \to G_i + \mathrm{i}(\Lambda - \Lambda^\dagger)$ with Λ a chiral superfield that serves as gauge parameter. We see that a shift of gauge parameter $\Lambda \to \Lambda + \zeta$ with ζ a real number leaves G_i invariant, so the original theory has this translation symmetry. If the symmetry remains when quantum correction enters, then it would prevent chiral superfield Λ from appearing in $\mathscr{B}$ and thus $\mathscr{B}_E$ cannot depend on G_i. On the contrary, if this translation symmetry is broken, then $\mathscr{B}_E$ can feel the existence of $\mathscr{A}$ through its dependence on Λ. In fact, this translation symmetry is only broken nonperturbatively, therefore $\mathscr{B}_E$ can have no G_i-dependence in perturbation theory, which we are assuming in the current proof. Then we have $\mathscr{B}_E = \mathscr{B}_E(\Phi, W_\alpha, T; L_i)$.

Additionally, there are two important symmetries in cut-off theory (7). One is a $U(1)$ R-symmetry, the other is the translation symmetry of T, $T \to T + \xi$. The latter translation is really a symmetry because the original Lagrangian (7) transforms into a spacetime total derivative. Therefore, this symmetry requires that T cannot appear in effective Lagrangian (8) except in its original form. So now we have,

$$\mathscr{B}_E(\Phi, W_\alpha, T; L_i) = T\,\mathrm{tr}\, W^\alpha W_\alpha \mathscr{B}_{E1} + \mathscr{B}_{E2}(\Phi, W_\alpha; L_i), \tag{9}$$

where $\mathscr{B}_{E1}$ is a function of scale E only.

To make use of R-symmetry, we perform a diagrammatic analysis. Consider an ordinary Feynman diagram (not super-diagram) containing E_V external “gaugino” lines, I_V internal V-lines, and X_m pure gauge vertices with $m \geq 3$ V-lines, Y_{mr} vertices with $m \geq r$ V-lines from terms with r factors of W_α in $\mathscr{B}(\Phi, W)$, as well as Z_m vertices with $m \geq 1$ V-lines from $\Phi^\dagger e^{2V}\Phi$. Then we have the following relation,

$$E_V + 2I_V = \sum_{m\geq 3} mX_m + \sum_r \sum_{m\geq r} mY_{mr} + \sum_m mZ_m. \tag{10}$$

Now let’s see the R-charge of this diagram. We require Φ, V, and T to be R-neutral. Then, since θ^α and $\bar\theta_{\dot\alpha}$ have R-charge $+1$ and -1, respectively (thus $\mathrm{d}^2\theta$ has R-charge -2), we see that W_α has R-charge $+1$ and $\mathscr{B}$ has R-charge $+2$. Thus L_r has R-charge

$2-r$, where r is the power of W_α in L_r-term. The diagram has E_V external gaugino lines, it must contribute to a term with E_V factors of W_α in $\mathscr{B}_E$ and the corresponding coefficient has R-charge $(2-E_V)$, since each W_α contains exactly one gaugino, and this is the reason we require the external V-lines to be gaugino lines rather than gauge boson lines. On the other hand, the R-charge of the coefficient resulted from the diagram above is totally contributed by Y_{mr} vertices, which has R-charge $(2-r)$. Therefore,

$$\sum_r \sum_{m\geq r}(2-r)Y_{mr} = 2-E_V. \tag{11}$$

Then, we can use the two equalities above to find the number of factors of τ in the given diagram to be,

$$\sum_{m\geq 3} X_m - I_V = 1 - \frac{1}{2}\bigg[\sum_{m\geq 3}(m-2)X_m + \sum_r\sum_{m\geq r}(2-r+m)Y_{mr} + \sum_m mZ_m\bigg]. \tag{12}$$

The quantity in the square bracket is semi-positive definite, thus the number of factors of τ is limited. When it contains 1 factors of τ, the possible numbers of vertices can be a) $X_3=2$ and all others vanish; b) $X_4=1$ and all others vanish; c) $Y_{m=r,r}=1$ and all others vanish; d) $Z_2=1$ and all others vanish; e) $X_3=1$, $Z_1=1$, and all others vanish. Now, the cases a) and b) are just 1-loop gauge corrections to $W^\alpha W_\alpha$ term, case d) is just the matter corrections to $W^\alpha W_\alpha$ term. These three cases combine to give 1-loop running gauge coupling τ_E (6). Furthermore, case c) gives tree diagrams that reproduces corresponding terms in $\mathscr{B}(\Phi, W_\alpha)$, and finally, case e) is not 1PI and thus does not contribute. It can also be seen from the equality above that the quantity in the square bracket vanishes only when $X_m, Y_{mr}, Z_m = 0$, which leaves nothing. Therefore, we conclude that $\mathscr{B}_E$ cannot depend on τ, thus $\mathscr{B}_{E1}=0$ and $\mathscr{B}_{E2} = \mathscr{B}(\Phi, W_\alpha) + \frac{1}{16\pi ik}(\tau_E - \tau)\,\mathrm{tr}\,W^\alpha W_\alpha$, namely the tree level result $\mathscr{B}(\Phi, W_\alpha)$, plus the one-loop correction to the gauge coupling τ, and this finishes the proof.

Wess-Zumino Superpotential. Although the theorem proved above can be directly applied to Wess-Zumino model with simple superpotential $\mathcal{W}(\Phi) = m\Phi^2 + g\Phi^3$, the non-renormalization of this superpotential can be proved with holomorphy argument more directly, as described in Seiberg's original paper [6]. In this case, we have a $U(1)\otimes U(1)_R$ symmetry in tree level superpotential, by assign Φ, m, and g the $U(1)\otimes U(1)_R$-charge $(1,1)$, $(-2,0)$, and $(-3,-1)$, respectively. Then, assuming this symmetry is not broken at any finite order in perturbation theory, the superpotential at energy scale $E \leq \Lambda_c$ must take the form $\mathcal{W}_E = m\Phi^2\mathscr{B}(g\Phi/m)$, where $\mathscr{B}$ is an arbitrary function of the indicated combination. Now expand the function $\mathscr{B}$ as a Laurent series, $\mathscr{B}(t) = \sum b_n t^n$, and consider the weak coupling limit. The superpotential should not be singular when $\lambda \to 0$ and $\mu \to 0$ requires that $n \geq 0$ and $n \leq 1$, thus $\mathscr{B} = b_0 + b_1 t$. The coefficient b_0 and b_1 can be determined to be 1 by requiring $\mathcal{W}_E \to \mathcal{W}$ when $m, g \to 0$ altogether. Thus we conclude that $\mathcal{W}_E = \mathcal{W}$.

3. Beta Function of Non-Abelian Gauge Theory

3.1 Instanton calculus and NSVZ β function

In this subsection we introduces the basics of instanton solution in supersymmetric gauge theory. For simplicity we take $N=1$ super Yang-Mills theory with gauge group $SU(2)$ as an example.

It is well known that instanton is a classical solution of gauge field in 4 dimensional Euclidean space satisfying the self-dual condition. Let the field strength be F_{mn}. Then the self-dual condition takes the form $F_{mn} = \pm\widetilde{F}_{mn}$ with $\widetilde{F}_{mn} = \frac{1}{2}\epsilon_{mnpq}F_{pq}$. The simplest nontrivial solution, the BPST instanton, is given by,

$$A_m \propto -\frac{\bar{\sigma}_{mn}(x-x_0)_n}{(x-x_0)^2+\rho^2}, \tag{13}$$

where the symbol $\bar{\sigma}_{mn}$ in Euclidean space is self-dual, x_0 and ρ mark the position and size of the instanton. It can be checked that this solution has winding number $k=+1$ where $k = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\, F_{mn}\widetilde{F}_{mn}$.

However, we would encounter a problem if we try to generalize this solution to $N=1$ supersymmetric theory directly. The problem arises from the fact that the Euclidean space does not admit a real spinor representation, while the $N=1$ supersymmetry in Minkowski spacetime is generated by a pair of Weyl spinor Q_α and $\overline{Q}_{\dot{\alpha}}$, which are complex conjugates of each other, and thus do form a real Majorana spinor. In Euclidean space, the left and right spinors are not related by complex conjugate, thus we simply have no $N=1$ supersymmetry in 4 dimensional Euclidean space which can be regarded as direct generalization of Minkowski one.

The solution to this problem lies in the fact that whenever the physical effects of instanton are concerned with, we can always treat the solution as a Wick rotation from Minkowski space. In pure bosonic theory, it is possible to rotate both coordinates and fields, so that the results in Euclidean space can be put into a real action. On the contrary, the fermion kinetic term in Minkowski space, when rotated to Euclidean space, would ceases to be real. Therefore, we may adopt the prescription that only the spacetime coordinates are wick rotated, while the fields are left intact. In this way, we can reformulate the physics of instanton in a consistent way.

With Minkowski signature, the self-dual condition becomes $\widetilde{F}_{mn} = \pm\mathrm{i}F_{mn}$, and we will all the condition with positive and negative signs the selfdual and antiselfdual, respectively. Then, with our convention, we find that σ_{mn} is antiselfdual and $\bar{\sigma}_{mn}$ is selfdual. Thus, we can directly write down a solution in Minkowski space which is similar to BPST instanton in Euclidean space, as follows,

$$A_m = -\frac{2\mathrm{i}\bar{\sigma}_{mn}(x-x_0)^n}{(x-x_0)^2+\rho^2}, \tag{14}$$

while the field strength is given by,

$$F_{mn} = \partial_m A_n - \partial_n A_m + \mathrm{i}[A_m, A_n] = \frac{4\mathrm{i}\bar{\sigma}_{mn}\rho^2}{[(x-x_0)^2+\rho^2]^2}. \tag{15}$$

3.2 Effective action: Wilsonian vs. 1PI

3.3 Rescaling anomaly of vector superfield

4. Nonperturbative Corrections to Effective Action

In last section we learned that the superpotential of an $N = 1$ susy theory receives no quantum correction at any finite order in perturbation theory, while the gauge coupling are renormalized only at 1-loop. Now we are going to study the nonperturbative corrections to this results. This is important because non-Abelian gauge theories usually exhibit asymptotic freedom in the UV, which means that the theory may become strongly coupled at IR, in which case the perturbation theory fail to work.

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