Instantons

(Preliminary version)

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Abstract

In this talk I would like to introduce some basics of instanton in pure $SU(N)$ gauge theory. After addressing some general (especially topological) properties of instanton solutions, we will construct the BPST instanton explicitly for $SU(2)$ theory and study their imbeddings in $SU(N)$ group. Then we will pay much attention to moduli and zero modes, which play the central roles in understanding the nature of instantons. In particular, we will show the one-to-one correspondence between moduli and zero modes, and count the d.o.f. of the latter by means of the index theorem. We will also compare the zero-mode counting here with the one in bc ghost system of string theory. Then I will try to outline the calculations of vacuum energy and the beta function in super Yang-Mills theory with instanton background. To achieve this goal, a large portion of the talk will be devoted to calculating the moduli metric and also the path-integral measure.

1 Introduction

In this note we will mainly focus on the $SU(N)$ Yang-Mills theory in 4d spacetime with flat Euclidean metric $g_{\mu\nu} = \eta_{\mu\nu}$. The action is given by

$$S = -\frac{1}{2g^2}\int \mathrm{d}^4x \ \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}, \tag{1}$$

with $F_{\mu\nu} = F^a_{\mu\nu}T_a$ and T_a are generators of $SU(N)$ in their fundamental representation.

Our goal is to find instanton solutions. By definition, the field configuration of an instanton makes the action finite. Therefore we can firstly restrict ourselves to those configurations with finite action. In 4 dimensions, this means that the corresponding field strength should go to zero faster than $|x|^{-2}$ as x goes to infinity.

Asymptotical approaching to Pure Gauge. Instantons are solutions of equations of motion $\mathrm{D}_\mu F_{\mu\nu} = 0$ that leave the action finite. Another very important point about instanton is that it is topologically nontrivial, in the sense that the winding number k, defined by

$$k = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\, \mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu}, \tag{2}$$

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is nonzero for instantons. Therefore the gauge potential for an instanton should reduce to zero fast enough as x approaches to infinity to guarantee that the action remains finite, while it should reduce to zero not too fast to leave a zero winding number. At a first sight this seems paradoxical by direct counting of the asymptotic power behavior. Nevertheless, we will now show that $F_{\mu\nu}(x)$ goes to zero faster than x^{-2} does not necessarily require the gauge potential to approach zero at infinity. Actually, we have the following:

- *Lemma 1: The field strength approaches to 0 as $x \to \infty$ implies that associated gauge potential approaches a pure gauge as $x \to \infty$, namely, $A_\mu \to U\partial_\mu U^{-1}$.*

To prove this lemma, it is enough to show that gauge fields are pure gauge when the curvature $F_{\mu\nu} = 0$, which can be most easily proved by using inverse-path-ordered Wilson lines[1]. Since $F_{\mu\nu} = 0$, the value of a Wilson line

$$\bar{W} = \bar{\mathrm{P}} \exp\left[\int_{x'=x}^{x'=y} A_\mu(x')\mathrm{d}x'^\mu\right] \tag{3}$$

depends only on its two endpoints but not on the path, since we know that the value of a Wilson loop is equals to an integration of curvature over the area enclosed by the loop, which in turn vanishes identically. Thus we may write the Wilson line as $\bar{W} = \bar{W}(x, y)$. Now, if we take the derivative of $\bar{W}$ with respect to y from the right side, we immediately get

$$\partial_\mu^{(y)}\bar{W}(x, y) = \bar{W}(x, y)A_\mu(y), \tag{4}$$

which implies directly that $A_\mu(y)$ is a pure gauge:

$$A_\mu(x) = \bar{W}^{-1}(y)\partial_\mu \bar{W}(y), \tag{5}$$

where we have suppressed the dependence of $W(x, y)$ on x. This finishes the proof of Lemma 1.

Topological nontriviality. Now we will show that an asymptotically pure-gauge-like potential A_μ indeed yields a nonzero winding number, and thus can be classified by the latter. We will calculate the winding number directly. To begin with, we observe that $\mathrm{tr}\,\tilde{F}_{\mu\nu}F_{\mu\nu}$ is a total derivative of a current known as Chern-Simons class:

$$\begin{aligned}\mathrm{tr}\,\tilde{F}_{\mu\nu}F_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{tr}\,\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 2\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\Big(\partial_\mu A_\nu + 2(\partial_\mu A_\nu)A_\rho A_\nu + A_\mu A_\nu A_\rho A_\sigma\Big)\\ &= 2\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,\partial_\mu\Big(A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma\Big).\end{aligned} \tag{6}$$

Then the winding number defined above reduces to a surface integral at infinity, by Stokes' theorem:

$$\begin{aligned}k &= -\frac{1}{8\pi^2}\oint_{S^3}\mathrm{d}\Omega_\mu\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\Big(A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma\Big) = \frac{1}{24\pi^2}\oint_{S^3}\mathrm{d}\Omega_\mu\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,(A_\nu A_\rho A_\sigma)\\ &= \frac{1}{24\pi^2}\oint_{S^3}\mathrm{d}\Omega_\mu\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\Big[(U^{-1}\partial_\nu U)(U^{-1}\partial_\rho U)(U^{-1}\partial_\sigma U)\Big].\end{aligned} \tag{7}$$

[1] By inverse path order we mean that the operator $A(x(t))$ with larger parameter t comes to the right. We denote all such expressions, including the inverse-path-order sign $\bar{\mathrm{P}}$ itself, with bars to highlight the difference from the usual convention.

The second equality holds due to $F_{\mu\nu} = 2(\partial_{[\mu}A_{\nu]} + A_{[\mu}A_{\nu]}) = 0$.

As an example, let us consider the fundamental map,

$$U(x) = \mathrm{i}x_\mu\sigma_\mu/\sqrt{x^2}, \qquad U^{-1}(x) = -\mathrm{i}x_\mu\sigma_\mu/\sqrt{x^2}. \tag{8}$$

Then it is easy to calculate that

$$U^{-1}\partial_\mu U =$$

To finish the surface integral, our strategy will be to change the integral variable from spatial coordinates to group parameters. For definiteness, we illustrate this with a specific example of $SU(2)$, in which case the space of group elements has dimension 3 and can be parameterized by 3 continuous variables $\xi^i(x)$ $(i = 1, 2, 3)$. Then the integrand can be rewritten as

$$\begin{aligned}
&\mathrm{d}\Omega_\mu\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\left[(U^{-1}\partial_\nu U)(U^{-1}\partial_\rho U)(U^{-1}\partial_\sigma U)\right]\\
&= \left(\tfrac{1}{6}\epsilon_{\mu\alpha\beta\gamma}\mathrm{d}x_\alpha\mathrm{d}x_\beta\mathrm{d}x_\gamma\right)\epsilon_{\mu\nu\rho\sigma}(\partial_\nu\xi^i)(\partial_\rho\xi^j)(\partial_\sigma\xi^k)\,\mathrm{tr}\left[(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U)\right]\\
&= \mathrm{d}\xi^{[i}\mathrm{d}\xi^j\mathrm{d}\xi^{k]}\,\mathrm{tr}\left[(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U)\right] = \mathrm{d}^3\xi\,\epsilon^{ijk}\,\mathrm{tr}\left[(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U)\right]\\
&= \mathrm{d}^3\xi\,\epsilon^{ijk}e_i^a e_j^b e_k^c\,\mathrm{tr}\,(T_aT_bT_c) = \mathrm{d}^3\xi\,(\det e)\,\mathrm{tr}\left(\epsilon^{abc}T_aT_bT_c\right) = -\tfrac{3}{2}\mathrm{d}^3\xi(\det e),
\end{aligned} \tag{9}$$

where the following relations have been used in turn:

$$\begin{aligned}
&\mathrm{d}\Omega_\mu = \tfrac{1}{6}\epsilon_{\mu\alpha\beta\gamma}\mathrm{d}x_\alpha\mathrm{d}x_\beta\mathrm{d}x_\gamma,\\
&\tfrac{1}{6}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} = \delta^{\nu\rho\sigma}_{[\alpha\beta\gamma]}\\
&\mathrm{d}\xi^{[i}\mathrm{d}\xi^j\mathrm{d}\xi^{k]} = \epsilon^{ijk}\mathrm{d}^3\xi,\\
&U^{-1}\partial_i U = e_i^a(\xi)T_a.
\end{aligned}$$

The group vielbein e_i^a have been used since the group metric is not unity in our convention. Therefore we have then

$$k = -\frac{1}{16\pi^2}\int \mathrm{d}^3\xi\ \det e. \tag{10}$$

(Anti-)Selfdual condition. Now we show that, in a given topological class, the self-dual or anti-self-dual field configurations must minimize the action. Therefore it must be an instanton solution provided that it leaves the action finite. (Yet not all instantons have to be self-dual or anti-self-dual.)

The (anti-)self-dual condition says that $F_{\mu\nu} = \pm{}^*F_{\mu\nu}$, (+ for self-dual and − for anti-self-dual) where the dual field strength is defined to be ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}$.

This conclusion can be easily reached by completing the square. That is,

$$\begin{aligned}
S &= -\frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{tr}\,F^2 = -\frac{1}{4g^2}\int \mathrm{d}^4x\,\mathrm{tr}\left(F \pm^* F\right)^2 \mp \frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{tr}\,F^*F\\
&\geq \mp\frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{tr}\,F^*F = \frac{8\pi^2}{g^2}(\pm k).
\end{aligned} \tag{11}$$

In the following we study self-dual or anti-self-dual instantons in most cases.

Vanishing energy-momentum tensor. The self-dual and anti-self-dual configurations always have vanishing energy-momentum tensor, which is a direct consequence of the topological nature of the configuration. This can be most easily verified by rewritten the action into a generally covariant form. Note that the action in this case is proportional to

$$\int \mathrm{d}^4x\, \epsilon_{\mu\nu\rho\sigma}\, \mathrm{tr}\, F_{\mu\nu}F_{\rho\sigma}. \tag{12}$$

After transform this expression to a generally covariant form, we have $\mathrm{d}^4x \to \mathrm{d}^4x\sqrt{g}$ while $\epsilon_{\mu\nu\rho\sigma} \to \sqrt{g}^{-1}\epsilon_{\mu\nu\rho\sigma}$. Note that the field strength $F_{\mu\nu}$ remains unchanged. Therefore the action is independent of the spacetime metric. Then the energy-momentum tensor vanishes, since it is defined to be the functional variation of the action with respect to the metric.

2 Construction of Instanton Solutions

In this section we construct the $k=1$ instanton solution to $SU(2)$ gauge theory. This amounts to find solutions to selfduality or anti-selfduality equations.

The BPST instanton of $SU(2)$. The construction begins with the following ansatz for gauge potential:

$$A_\mu(x) = \alpha\sigma_{\mu\nu}\partial_\nu \log\phi(x^2). \tag{13}$$

Then the field strength and dual field strength read:

$$\begin{aligned}
F_{\mu\nu} =&\ \alpha\sigma_{\nu\rho}\partial_\mu\partial_\rho\log\phi - \alpha\sigma_{\mu\rho}\partial_\nu\partial_\rho\log\phi^2 + \alpha^2[\sigma_{\mu\rho},\sigma_{\nu\sigma}](\partial_\rho\log\phi)(\partial_\sigma\log\phi)\\
=&\ \big(\alpha\sigma_{\nu\rho}\partial_\mu\partial_\rho\log\phi - (\mu\leftrightarrow\nu)\big) + 2\alpha^2\big(\sigma_{\mu\sigma}(\partial_\nu\log\phi)(\partial_\sigma\log\phi) - (\mu\leftrightarrow\nu)\big)\\
&- 2\alpha^2\sigma_{\mu\nu}(\partial\log\phi)^2; \qquad (14)\\
{}^*F_{\mu\nu} =&\ \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}\\
=&\ \alpha\epsilon_{\mu\nu\rho\sigma}\sigma_{\nu\lambda}\partial_\rho\partial_\lambda\log\phi + 2\alpha^2\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\lambda}(\partial_\sigma\log\phi)(\partial_\lambda\log\phi) - \alpha^2\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}(\partial\log\phi)^2\\
=&\ \sigma_{\nu\rho}\big(\alpha\partial_\rho\partial_\mu\log\phi - 2\alpha^2(\partial_\rho\log\phi)(\partial_\mu\log\phi) - (\mu\leftrightarrow\nu)\big) + \sigma_{\mu\nu}(\alpha\partial^2\log\phi). \qquad (15)
\end{aligned}$$

where we have used the following relations:

$$\begin{aligned}
[\sigma_{\mu\nu},\sigma_{\rho\sigma}] =&- 2\big(\delta_{\mu\rho}\sigma_{\nu\sigma} + \delta_{\nu\sigma}\sigma_{\mu\rho} - \delta_{\mu\sigma}\sigma_{\nu\rho} - \delta_{\nu\rho}\sigma_{\mu\sigma}\big), \qquad (16)\\
\sigma_{\mu\nu} =&- \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}. \qquad (17)
\end{aligned}$$

The commutator is easy to understand since $\sigma_{\mu\nu}/2$ is nothing but the rotation generators in Euclidean space. Now the selfdual condition $F = *F$ gives rise to two equations, corresponding to the coefficients before $\sigma_{\nu\rho}$ and $\sigma_{\mu\nu}$ respectively. The former equation is a trivial identity while the latter gives

$$-2\alpha^2(\partial\log\phi)^2 = \alpha\partial^2\log\phi. \tag{18}$$

This equation can be further simplified by change the variable $\phi \to \phi^{1/2\alpha}$ to

$$\phi^{-1}\Box\phi = 0, \tag{19}$$

the solution of which is easy to find to be $\phi(x) = x^{-2}$, or, more generally,

$$\phi(x) = \frac{\rho^2}{(x-a)^2} + C. \tag{20}$$

Therefore the gauge potential

$$A_\mu(x) = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \log\left[C + \frac{\rho^2}{(x-a)^2}\right]. \tag{21}$$

The condition that $A_\mu(x)$ should vanish for large $|x|$ implies that C should be taken to be 1. Therefore we conclude that

$$A_\mu(x) = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \log\left[1 + \frac{\rho^2}{(x-a)^2}\right] = -\sigma_{\mu\nu}\frac{\rho^2(x-a)_\nu}{(x-a)^2[(x-a)^2+\rho^2].} \tag{22}$$

This is precisely the $k=1$ instanton solution of $SU(2)$ theory in the singular gauge. To bring it to a regular form, we take a singular gauge transformation to remove the singularity at $x=a$. That is, we take $U(x) = \mathrm{i}\bar\sigma_\mu x_\mu/\sqrt{x^2}$, then

$$U\big(\partial_\mu + A_\mu(x)\big)U^{-1} = (\partial_\mu U)U^{-1}\bigg(-1 + \frac{\rho^2}{(x-a)^2+\rho^2}\bigg) = -\bar\sigma_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2+\rho^2} \tag{23}$$

yields a regular solution, known as BPST instanton [1].

Here we have a remark. Note that the rotational symmetry group $SO(4)$ has the covering group $SU(2)_L\times SU(2)_R$. Therefore every $SO(4)$ tensor can be written in the form carrying $SU(2)_L\times SU(2)_R$ indices. In our construction above, the $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$ are two copies of rotation generators living in $SU(2)_L$ and $SU(2)_R$, respectively. Then we see that the regular solution identifies the gauge group $SU(2)$ with the $SU(2)_L$ part of the rotational symmetry and the singular solution identifies that with $SU(2)_R$. These identifications reveal the fact that the instanton solution breaks the whole product group $SU(2)_L\times SU(2)_R$ down to a diagonal $SU(2)$.

The BPST solution derived above given above has nonzero winding number. The field strength can be calculated as:

$$\begin{aligned}F_{\mu\nu} =& \partial_\mu\bigg[-\bar\sigma_{\nu\rho}\frac{(x-a)_\nu}{(x-a)^2+\rho^2}\bigg] - (\mu\leftrightarrow\nu) + \frac{(x-a)_\rho(x-a)_\sigma}{[(x-a)^2+\rho^2]^2}[\bar\sigma_{\mu\rho},\bar\sigma_{\nu\sigma}]\\ =&\bigg[\frac{\bar\sigma_{\mu\nu}}{(x-a)^2+\rho^2} + \frac{2(x-a)_\mu(x-a)_\rho}{[(x-a)^2+\rho^2]^2}\bar\sigma_{\nu\rho}\bigg] - (\mu\leftrightarrow\nu)\\ &- \frac{2}{[(x-a)^2+\rho^2]^2}\Big((x-a)^2\bar\sigma_{\mu\nu} - (x-a)_\rho(x-a)_\mu\bar\sigma_{\rho\nu} - (x-a)_\nu(x-a)_\rho\bar\sigma_{\mu\sigma}\Big)\\ =&\frac{2\rho^2\bar\sigma_{\mu\nu}}{[(x-a)^2+\rho^2]^2}.\end{aligned} \tag{24}$$

Then the winding number is

$$k = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\ \mathrm{tr}\, F^*_{\mu\nu}F_{\mu\nu} = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\,\frac{4\rho^4}{[(x-a)^2+\rho^2]^4}\,\mathrm{tr}\,\bar\sigma_{\mu\nu}\bar\sigma_{\mu\nu} = 1. \tag{25}$$

The trace of $\bar\sigma\bar\sigma$ can be worked out through $\{\bar\sigma_{\mu\nu},\bar\sigma_{\rho\sigma}\} = -2(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\alpha}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma})$.

It is easy to generalize the solution above to k-instantons, by taking

$$\phi(x) = 1 + \sum_{i=1}^{k} \frac{\rho_i^2}{(x-a_i)^2}, \qquad \text{('t Hooft)}$$

or, more generally,

$$\phi(x) = \sum_{i=1}^{k+1} \frac{\rho_i^2}{(x-a_i)^2}, \qquad \text{(Jackiw, Nohl and Rebbi).}$$

2.1 Collective Coordinates

Imbedding into $SU(N)$ Now that we have got the k-instanton solutions to $SU(2)$ gauge theory (though not complete), we can find the $SU(N)$ instantons by directly imbedding the $SU(2)$ solutions to $SU(N)$. More precisely, let $A_\mu^{SU(2)}$ be a $SU(2)$ $k=1$ instanton, then

$$A_\mu^{SU(N)} = \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix}. \tag{26}$$

This is obviously an $SU(N)$ instanton with $k=1$ since it gives a selfdual field strength and leave the action finite. Actually we can apply $SU(N)$ transformations to this solution to generate new instantons, except for those $SU(N)$ rotations that leave this solution invariant. These rotations consist of ones affecting zero elements only and ones commuting with this solution trivially. They form subgroups of $SU(N)$, namely $SU(N-2)$ and $U(1)$, respectively. Therefore, we conclude that for this embedding of $k=1$ $SU(2)$ instanton, we can apply $SU(N)/\big(SU(N-2)\times U(1)\big)$ to generate independent new solutions. This quotient group has dimension $(N^2-1)-((N-2)^2-1)-1 = 4N-5$. Together with the position and scale of $SU(2)$ instanton, we get $4N$ collective coordinates for $k=1$ $SU(N)$ instantons.

3 Counting the Zero Modes

In this section we begin an investigation of zero modes of gauge field in the instanton background. In particular, we will count the d.o.f. of zero modes by index theorem. Finally we will reach the conclusion that the number of independent zero modes does coincide with the number of moduli. For the instanton of $SU(N)$ theory with winding number k, the number of d.o.f. of zero modes is $4N|k|$, just the same as the result of ADHM construction, as can also be seen from the one-to-one correspondence between modulus and zero mode.

Moduli, zero modes, and symmetries. Firstly let us make a remark on the relation between moduli and zero modes. Actually the relation between zero modes and moduli can be quite easily understood from a intuitive point of view. Indeed, we recall that moduli characterize the d.o.f. of the solution to the equations of motion in a given topological class, and thus can be regarded as labels of different vacua with the same winding number; while the zero modes are gauge inequivalent solutions to the linearized equations of motion for fluctuations (not the background field). Therefore we may naturally view the zero modes as Goldstone modes. They appear in the theory simply due to the degeneracy of vacuum states, which breaks down some global symmetries spontaneously, and is in turn characterized by moduli.

This simple argument is justified in the simplest example of $k = 1$ solution to $SU(2)$ theory. In this case there are 8 collective coordinates associated with spacetime coordinates, a dilatation parameter, and 3 global gauge parameters, respectively. Then we may say that they break the translational symmetry, the global conformal symmetry, and the global gauge symmetry, respectively. Indeed 8 zero modes are generated in this case and can be regarded as Goldstone modes.

However, this is far from a rigorous argument, since we know that the conventional Goldstone theorem applies only to the spontaneous breaking of internal symmetry with spacetime Lorentz symmetry being manifest. When this is not the case, say, if the broken symmetry is not internal, or, the spacetime Lorentz symmetry is replaced with Euclidean symmetry, the usual one-to-one correspondence between Goldstone modes and broken symmetries does not hold any more. Thus one should be very careful when apply the arguments of spontaneous symmetry breaking to our case.

Constraint on moduli. The moduli space, by definition, is the space of all vacua associated with a given topological class. Let us restrict ourselves to its local property for the moment. That is, we consider the continuous parameters (collective coordinates) characterizing these vacua only. Then, slight varying these coordinates is equivalent to deforming the gauge potential by a small quantity δA_μ. By definition, this variation should also satisfies the selfdual or antiselfdual condition. This gives

$$f_{\mu\nu} \equiv \mathrm{D}_\mu \delta A_\nu - \mathrm{D}_\nu \delta A_\mu = {}^*(\mathrm{D}_\mu \delta A_\nu - \mathrm{D}_\nu \delta A_\mu). \tag{27}$$

On the other hand, moduli describes gauge inequivalent vacua, thus the deformation δA_μ should not be a gauge variation. That is, it should be orthogonal to all gauge transformations,

$$(\mathrm{D}_\mu \Lambda, \delta A_\mu) \equiv \int \mathrm{d}^4 x \ \mathrm{tr}\left[(\mathrm{D}_\mu \Lambda)\delta A_\mu\right] = 0. \tag{28}$$

Performing the integration by parts, we see that this orthogonal condition is nothing but the usual "gauge condition" $\mathrm{D}_\mu \delta A_\mu = 0$. Then we conclude that collective coordinates are those deformations of gauge potential which satisfy selfdual (antiselfdual) condition and the gauge condition.

Zero mode equations. Now let us begin the study of zero modes. In particular, we will show that zero modes do satisfy the selfdual (antiselfdual) condition as well as the gauge condition.

Our strategy is write down the linearized equations of motion for these fluctuating fields, which is of course a differential equation with a differential operator. Zero modes can be naturally identified with the points in the kernel of this operator. Then, we will relate the kernel of the operator to the kernel of a Dirac operator, and the d.o.f of latter can be easily counted by calculating the familiar chiral anomaly.

Firstly, we separate the gauge potential A_μ into two parts:

$$A_\mu = A_\mu^{\mathrm{cl}} + \mathcal{A}_\mu, \tag{29}$$

with A_μ^{cl} the background field satisfying the classical equations of motion and will be taken to be instanton solution in our treatment, and $\mathcal{A}_\mu$ are quantum fluctuations. The expanding for the curvature $F_{\mu\nu}$ goes accordingly, as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu(A_\nu^{\rm cl}+\mathcal{A}_\nu)-\partial_\nu(A_\mu^{\rm cl}+\mathcal{A}_\mu)+\big[(A_\mu^{\rm cl}+\mathcal{A}_\mu),(A_\nu^{\rm cl}+\mathcal{A}_\nu)\big] \\ &= \big(\partial_\mu A_\nu^{\rm cl}-\partial_\nu A_\mu^{\rm cl}+[A_\mu^{\rm cl},A_\nu^{\rm cl}]\big)+\big(\partial_\mu\mathcal{A}_\nu+[A_\mu^{\rm cl},\mathcal{A}_\nu]\big)-\big(\partial_\nu\mathcal{A}_\mu+[A_\nu^{\rm cl},\mathcal{A}_\mu]\big)+[\mathcal{A}_\mu,\mathcal{A}_\nu] \\ &\equiv F_{\mu\nu}^{\rm cl}+{\rm D}_\mu^{\rm cl}\mathcal{A}_\nu-{\rm D}_\nu^{\rm cl}\mathcal{A}_\mu+[\mathcal{A}_\mu,\mathcal{A}_\nu]\equiv F_{\mu\nu}^{\rm cl}+\mathcal{F}_{\mu\nu}+[\mathcal{A}_\mu,\mathcal{A}_\nu]. \end{aligned} \tag{30}$$

Then the Lagrangian, expanded to quadratic order in $\mathcal{A}_\mu$, reads

$$\mathcal{L}=\frac{1}{4}(F_{\mu\nu}^a)^2=\frac{1}{4}(F_{\mu\nu}^{a,{\rm cl}})^2+\frac{1}{4}(\mathcal{F}_{\mu\nu}^a)^2+\frac{1}{2}f_{abc}F_{\mu\nu}^{a,{\rm cl}}\mathcal{A}_\mu^b\mathcal{A}_\nu^c+\mathcal{O}(\mathcal{A}^3). \tag{31}$$

Gauge fixing term is of course needed:

$$\mathcal{L}_{\rm fix}=\frac{1}{2}({\rm D}_\mu^{\rm cl}\mathcal{A}_\mu^a)^2. \tag{32}$$

Combing this term into the original Lagrangian, we get

$$\mathcal{L}^{(2)}=\frac{1}{4}(\mathcal{F}_{\mu\nu}^a)^2+\frac{1}{2}f_{abc}F_{\mu\nu}^{a,{\rm cl}}\mathcal{A}_\mu^b\mathcal{A}_\nu^c+\frac{1}{2}({\rm D}_\mu^{\rm cl}\mathcal{A}_\mu^a)^2. \tag{33}$$

Now we will show that this Lagrangian can be put into the following form:

$$\mathcal{L}^{(2)}=\frac{1}{8}(\mathcal{F}_{\mu\nu}^a+^*\mathcal{F}_{\mu\nu}^a)^2+\frac{1}{2}({\rm D}_\mu^{\rm cl}\mathcal{A}_\mu^a)^2. \tag{34}$$

To show this, we firstly note that $(^*\mathcal{F}_{\mu\nu}^a)^2=(\mathcal{F}_{\mu\nu}^a)^2$. Therefore the first term in (33) can be rewritten as $\frac{1}{8}(^*\mathcal{F}_{\mu\nu}^a)^2+\frac{1}{8}(\mathcal{F}_{\mu\nu}^a)^2$. The second term in (33) can be most easily understood in matrix form:

$$\begin{aligned} \tfrac{1}{2}f_{abc}F_{\mu\nu}^{a,{\rm cl}}\mathcal{A}_\mu^b\mathcal{A}_\nu^c &= -\ {\rm tr}\,\big(F_{\mu\nu}^{\rm cl}\mathcal{A}_\mu\mathcal{A}_\nu\big)=\mp\epsilon_{\mu\nu\rho\sigma}\,{\rm tr}\,\big(F_{\rho\sigma}^{\rm cl}\mathcal{A}_\mu\mathcal{A}_\nu\big)=\mp\epsilon_{\mu\nu\rho\sigma}\,{\rm tr}\,\big(\mathcal{A}_\nu F_{\rho\sigma}^{\rm cl}\mathcal{A}_\mu\big) \\ &= \mp\,\epsilon_{\mu\nu\rho\sigma}\,{\rm tr}\,\big(\mathcal{A}_\nu[{\rm D}_\rho^{\rm cl},{\rm D}_\sigma^{\rm cl}]\mathcal{A}_\mu\big)=\mp2\epsilon_{\mu\nu\rho\sigma}\,{\rm tr}\,\big(\mathcal{A}_\nu{\rm D}_\rho^{\rm cl}{\rm D}_\sigma^{\rm cl}\mathcal{A}_\mu\big) \\ &= \pm\,2\epsilon_{\mu\nu\rho\sigma}\,{\rm tr}\,\big[({\rm D}_\rho^{\rm cl}\mathcal{A}_\nu)({\rm D}_\sigma^{\rm cl}\mathcal{A}_\mu)\big]=\pm\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,{\rm tr}\,\big(\mathcal{F}_{\rho\nu}\mathcal{F}_{\sigma\mu}\big) \\ &= \pm\,\tfrac{1}{4}\mathcal{F}_{\mu\nu}^a\,{}^*\mathcal{F}_{\mu\nu a}. \end{aligned} \tag{35}$$

Then (34) follows directly from (33).

Now if a fluctuating field $\mathcal{A}_\mu$ solves the linearized equations above, which can be symbolically written as $\nabla\mathcal{A}=0$ with some differential operator ∇, then it must make the action vanish, $\int {\rm d}^4x\,\mathcal{A}\nabla\mathcal{A}=0$. Since the Lagrangian is a sum of two squares, this means each square should vanish identically. Therefore we see that the solution of the linearized equation does satisfy the (anti)selfdual condition and the gauge condition. That is, each zero mode gives a modulus. Conversely, each modulus λ gives a zero mode $\mathcal{A}_\mu$ via $\mathcal{A}_\mu=\delta\lambda\frac{A_\mu}{\delta\lambda}$.

3.1 From Laplace Operator to Dirac Operator

Now we see that the equations satisfied by zero modes are provided by the (anti)selfdual condition and the gauge condition, which can be put into the following form:

$$\bar\sigma_{\mu\nu}{\rm D}_\mu^{\rm cl}\mathcal{A}_\nu=0, \qquad {\rm D}_\mu^{\rm cl}\mathcal{A}_\mu=0. \tag{36}$$

These two equations can be further rewritten into a more compact form,

$$\bar{\sigma}_\mu \sigma_\nu \mathrm{D}_\mu \mathcal{A}_\nu = 0. \tag{37}$$

From now on we will drop the superscript 'cl'. As mentioned previously, we will relate the zero mode counting of this Laplace operator to that of the Dirac operator. To achieve this, let us introduce the two-component notations, with

$$\bar{\not{D}} \equiv (\bar{\sigma}_\mu)_{\alpha'\beta}\mathrm{D}_\mu = \bar{\mathrm{D}}_{\alpha'\beta}; \qquad (\sigma_\nu)^{\alpha\beta'}\mathcal{A}_\nu = \mathbf{A}^{\alpha\beta'}. \tag{38}$$

Then the equation for zero modes can be written as

$$\bar{\not{D}}\mathcal{A} = \bar{\mathrm{D}}_{\alpha'\beta}\mathbf{A}^{\beta\gamma'} = \partial_{\alpha'\beta}\mathbf{A}^{\beta\gamma'} + [A^{\mathrm{cl}}_{\alpha'\beta}, \mathbf{A}^{\beta\gamma'}] = 0. \tag{39}$$

Note that $\mathbf{A}^{\alpha\beta'}$ is a 2×2 matrix given by

$$\mathbf{A} = \begin{pmatrix} \mathcal{A}_3 + \mathrm{i}\mathcal{A}_4 & \mathcal{A}_1 - \mathrm{i}\mathcal{A}_2 \\ \mathcal{A}_1 + \mathrm{i}\mathcal{A}_2 & -\mathcal{A}_3 + \mathrm{i}\mathcal{A}_4 \end{pmatrix} \equiv \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix}, \tag{40}$$

therefore each column of $\mathbf{A}$, as a two-component spinor, satisfies the zero mode equation respectively. That is, if we introduce $\lambda = \binom{a}{b}$, then $\mathbf{A} = (\lambda, \mathrm{i}\sigma^2\lambda^*)$ and $\bar{\not{D}}\lambda = 0$. Then, for each given λ, λ and $\mathrm{i}\sigma^2\lambda^*$ provide two independent deformations of the anti-instanton. Furthermore, $\mathbf{A}$ multiplied by a factor i, namely $\mathrm{i}\mathbf{A} = (\mathrm{i}\lambda, -\sigma^2\lambda)$, gives additional independent deformation. This is because the deformation $\mathcal{A}$ itself is a real function. Thus each independent complex solution should be counted twice. Therefore, we conclude that for each solution λ to the equation $\bar{\not{D}}\lambda = 0$, there are four associated independent solutions $\mathbf{A}$, given by $(\lambda, \mathrm{i}\sigma^2\lambda^*)$, $(\mathrm{i}\sigma^2\lambda^*, -\lambda)$, $(\mathrm{i}\lambda, -\sigma^2\lambda^*)$ and $(-\sigma^2\lambda^*, -\mathrm{i}\lambda)$.

Zero modes of Dirac operator. Now we recall that a 4-component massless Dirac fermion ψ satisfies the equation of motion $\gamma_\mu\mathcal{D}_\mu\phi = \not{\mathcal{D}}\psi = 0$, and can be decomposed into two 2-component Weyl spinors, $\psi = \binom{\lambda^\alpha}{\bar{\chi}_{\alpha'}}$. We further note that

$$\gamma^\mu = \begin{pmatrix} 0 & -\mathrm{i}\sigma^{\mu\alpha\beta'} \\ \mathrm{i}\bar{\sigma}^\mu_{\alpha'\beta} & 0 \end{pmatrix},$$

and $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$. Then the Dirac equation becomes $\bar{\not{D}}\lambda = 0$ and $\not{D}\bar{\chi} = 0$. Now we're going to show that with an anti-instanton background, $\not{D}$ has zero modes (namely, normalizable solutions) while $\bar{\not{D}}$ has not. The argument for $\not{D}\bar{\chi} = 0$ has zero solutions only can be summarized as:

$$\boxed{\not{D}\bar{\chi} = 0 \;\Rightarrow\; \mathrm{D}^2\bar{\chi} = 0 \;\Rightarrow\; \mathrm{D}_\mu\bar{\chi} = 0 \;\Rightarrow\; \partial_\mu\chi = 0 \;\Rightarrow\; \chi = 0,}$$

provided that χ is squared integrable. In detail, the strategy for the first "⇒" is to consider another two operators, namely $\bar{\not{D}}\not{D}$ and $\not{D}\bar{\not{D}}$. Then it is easy to see that $\ker\not{D} \subset \ker\bar{\not{D}}\not{D}$ and $\ker\bar{\not{D}} \in \ker\not{D}\bar{\not{D}}$. Now,

$$\bar{\not{D}}\not{D} = \bar{\sigma}_\mu\sigma_\nu\mathrm{D}_\mu\mathrm{D}_\nu = (\delta_{\mu\nu} + \bar{\sigma}_{\mu\nu})\mathrm{D}_\mu\mathrm{D}_\nu = \mathrm{D}^2 + \tfrac{1}{2}\bar{\sigma}_{\mu\nu}F_{\mu\nu},$$

where the second term vanishes for antiselfdual field strength (since $\bar{\sigma}_{\mu\nu}$ is selfdual):

$$\bar{\sigma}_{\mu\nu}F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}F_{\mu\nu} = -\bar{\sigma}_{\rho\sigma}F_{\rho\sigma} = 0. \tag{41}$$

Therefore the equation $\not{\mathrm{D}}\bar{\chi} = 0$ implies that $\mathrm{D}^2\bar{\chi} = 0$.

To justify the second "$\Rightarrow$", we multiply the equation above by χ^* from left, and integrate, to get

$$\int \mathrm{d}^4x\,|\mathrm{D}_\mu\bar{\chi}|^2 = 0.$$

Here the integration by parts has been done where the fact that $\bar{\chi}$ goes to zero fast enough at infinity is used. Then $\mathrm{D}_\mu\bar{\chi} = 0$.

The third "$\Rightarrow$" can be shown by noticing that $F_{\mu\nu}\bar{\chi} = [\mathrm{D}_\mu, \mathrm{D}_\nu]\bar{\chi}$. That is, $F^a_{\mu\nu}T^a\bar{\chi} = 0$. Now, the antiselfdual field strength $F_{\mu\nu}$ is proportional to antiselfdual 't Hooft tensor $\bar{\eta}_{a\mu\nu}$. But we know that $\eta_{a\mu\nu}\eta_{b\mu\nu} \propto \delta_{ab}$, therefore we have $T^a\bar{\chi} = 0$ for all T^a. Then the covariant derivative reduces to partial derivative, $\partial\bar{\chi} = 0$. For an integrable $\bar{\chi}$, it can have vanishing solution only.

The index of Dirac operator. The result above tells that counting the zero modes of $\bar{\not{\mathrm{D}}}$ ($\not{\mathrm{D}}$) for anti-instanton (instanton) solution is equivalent to counting the zero modes of the dirac operator $\not{\mathcal{D}}$. But the latter can be easily done with the aid of the index theorem. Note that the analytic index of the dirac operator is defined by the difference between the number of positive and negative chirality zero modes,

$$\mathrm{Ind}\not{\mathcal{D}} = n_+ - n_-, \tag{42}$$

where the sign of chirality is defined to be the eigenvalue of γ^5. Recall that γ^5 now has the form $\mathrm{diag}(1, -1)$, then we see that the positively chiral zero modes are the zero modes of $\bar{\not{\mathrm{D}}}$ while the negatively chiral zero modes are zero modes of $\not{\mathrm{D}}$. That is,

$$\mathrm{Ind}\not{\mathcal{D}} = \ker\bar{\not{\mathrm{D}}} - \ker\not{\mathrm{D}}. \tag{43}$$

For anti-instanton background, $\not{\mathrm{D}}$ has no zero mode, as shown above. Therefore, the index of $\not{\mathcal{D}}$ is simply equal to the number of zero modes of $\bar{\not{\mathrm{D}}}$.

Now we can calculate $\ker\bar{\not{\mathrm{D}}}$ through calculating $\mathrm{Ind}\not{\mathcal{D}}$, which, in turn, can be done by calculating the chiral anomaly, by the famous AS index theorem. This issue is what we are all familiar with. Let me give a quick sketch.

4 Moduli Space: the Metric and the Measure

In the next section we will study the problem of doing path integral with instanton background. The main feature here is that we should perform the integral over zero modes separately. Therefore this section is devoted to a study of moduli space. We will construct the complete set of zero modes with some specific examples and calculate the metric in the moduli space, which turns out to be a very important quantity.

4.1 Bosonic Zero Modes

Firstly let us consider the bosonic modes. As was done in the previous section, we separate the gauge potential into its background value $A_\mu^{\rm cl}$ and its quantum fluctuation $\mathcal{A}_\mu$:

$$A_\mu = A_\mu^{\rm cl} + \mathcal{A}_\mu. \tag{44}$$

To perform quantization, we also add the gauge fixing term $-\frac{1}{g^2}\,\mathrm{tr}\,(\mathrm{D}_\mu^{\rm cl}\mathcal{A}_\mu)^2$ and the corresponding ghost term $-b^a[\mathrm{D}_\mu^{\rm cl}\mathrm{D}_\mu c]^a$. Then the action expanded to quadratic order in fluctuations, reads

$$S = \frac{8\pi^2}{g^2}|k| + \frac{1}{g^2}\,\mathrm{tr}\int \mathrm{d}^4x\,\big(\mathcal{A}_\mu M_{\mu\nu}\mathcal{A}_\nu + 2bM^{\rm gh}c\big), \tag{45}$$

with

$$M_{\mu\nu} = M_{\mu\nu}^{(1)} + M_{\mu\nu}^{(2)} = \big(\mathrm{D}^2\delta_{\mu\nu} - \mathrm{D}_\nu\mathrm{D}_\mu + F_{\mu\nu}\big) + \mathrm{D}_\mu\mathrm{D}_\nu = \mathrm{D}^2\delta_{\mu\nu} + 2F_{\mu\nu}, \tag{46}$$

where we have dropped the superscript 'cl', and $M^{(1)}$ is the operator from classical action in quadratic order while $M^{(2)}$ is from gauge fixing.

The fluctuation filed $\mathcal{A}_\mu$ contains zero modes, as we have studies extensively in previous sections. Let γ_i denote the collective coordinates, then zero modes have the form

$$Z_\mu^{(i)} = \frac{\partial A_\mu^{\rm cl}}{\partial\gamma_i} + \mathrm{D}_\mu^{\rm cl}\Lambda^i, \tag{47}$$

where the gauge transformation with gauge parameter Λ is necessary in order that Z_μ is in the background gauge, namely $\mathrm{D}_\mu^{\rm cl}Z_\mu^{(i)} = 0$. Indeed, $\partial A/\partial\gamma$ is a solution to $M^{(1)} = \partial^2 S/\partial A^{\rm cl}\partial A^{\rm cl}$, which can be seen by differentiating the equations of motion with respect to collective coordinates:

$$0 = \frac{\partial}{\partial\gamma_i}\frac{\delta S^{\rm cl}}{\delta A_\mu^{\rm cl}(x)} = \int \mathrm{d}^4y\,\frac{\delta^2 S^{\rm cl}}{\delta A_\mu^{\rm cl}(y)\delta A_\nu^{\rm cl}(x)}\frac{\partial A_\nu^{\rm cl}(y)}{\partial\gamma^i}. \tag{48}$$

The gauge term $\mathrm{D}_\mu\Lambda$ also satisfies the classical equations of motion, namely $M_{\mu\nu}^{(1)}(\mathrm{D}_\nu\Lambda^i) = 0$, which can be proved directly:

$$M_{\mu\nu}^{(1)}(\mathrm{D}_\nu\Lambda) = (\mathrm{D}^2\mathrm{D}_\mu - \mathrm{D}_\nu\mathrm{D}_\mu\mathrm{D}_\nu + F_{\mu\nu}\mathrm{D}_\nu)\Lambda = (\mathrm{D}_\nu F_{\nu\mu})\Lambda + F_{\nu\mu}\mathrm{D}_\nu\Lambda + F_{\mu\nu}\mathrm{D}_\nu\Lambda = 0, \tag{49}$$

since $[\mathrm{D}_\mu,\mathrm{D}_\nu] = F_{\mu\nu}$ and $\mathrm{D}_\nu F_{\nu\mu} = 0$ by classical equations of motion. As we will show below, these $Z_\mu^{(i)}$ are also normalizable, therefore they are indeed zero modes.

Metric in moduli space of $SU(2)$ $k = \pm1$ instantons. Now we construct a metric on moduli space with the zero modes defined above, as

$$U^{ij} \equiv \langle Z^{(i)}|Z^{(j)}\rangle \equiv -\frac{2}{g^2}\int \mathrm{d}^4x\;\mathrm{tr}\,Z_\mu^{(i)}Z_\mu^{(j)} = \frac{1}{g^2}\int \mathrm{d}^4x\,Z_\mu^{(i)a}Z_\mu^{(j)a}. \tag{50}$$

Let us evaluate this metric with the $SU(2)$ $k = -1$ instanton. Recall that in this case we have 8 independent collective coordinates, including 4 translational modes, 1 dilatational mode and 3 global gauge transformations. Firstly, for translational modes, the gauge parameter $\Lambda^{(\nu)}$ can be chosen to be $A_\nu^{\rm cl}$. That is,

$$Z_\mu^{(\nu)} = \frac{\partial A_\mu^{\rm cl}}{\partial x_0^\nu} + {\rm D}_\mu^{\rm cl}\mathcal{A}_\nu^{\rm cl} = -\partial_\nu A_\mu^{\rm cl} + {\rm D}_\mu^{\rm cl} A_\nu^{\rm cl} = F_{\mu\nu}^{\rm cl}. \tag{51}$$

The resulted zero modes $F_{\mu\nu}^{\rm cl}$ satisfies the background gauge condition obviously. The norm of these modes can also be obtained straightforwardly, as

$$U^{\mu\nu} = -\frac{2}{g^2}\int {\rm d}^4x \ {\rm tr}\, F_{\lambda\mu}^{\rm cl}F_{\lambda\nu}^{\rm cl} = \frac{8\pi^2|k|}{g^2} = S_{\rm cl}\delta^{\mu\nu}. \tag{52}$$

We have spelled out the k-dependence explicitly and this result holds for any k and any gauge group.

Then consider the dilatation mode in $k=-1$ $SU(2)$ instanton. This time, taking derivative with respect to dilatation parameter ρ does not shift the zero mode out of the background gauge. Therefore the corresponding gauge parameter Λ can be chosen to be 0. Then in singular gauge we have

$$Z_\mu^{(D)} = \frac{\partial}{\partial\rho}\left(\frac{-\rho^2\bar\sigma_{\mu\nu}x_\nu}{x^2(x^2+\rho^2)}\right) = -\frac{2\rho\bar\sigma_{\mu\nu}x_\nu}{(x^2+\rho^2)^2}. \tag{53}$$

The background gauge condition ${\rm D}_\mu^{\rm cl}Z_\mu^{(D)}$ is satisfied, which can be seen solely from the indices structure:

$${\rm D}_\mu^{\rm cl}Z_\mu^{(D)} = \partial_\mu Z_\mu^{(D)} + [A_\mu^{\rm cl}, Z_\mu^{(D)}] = \big((\cdots)\bar\sigma_{\mu\nu}\delta_{\mu\nu} + (\cdots)\bar\sigma_{\mu\nu}x_\mu x_\nu\big) + (\cdots)[\bar\sigma_{\mu\rho},\bar\sigma_{\mu\sigma}]x_\rho x_\sigma = 0. \tag{54}$$

Then, the norm is given by

$$U^{DD} = -\frac{2}{g^2}\int {\rm d}^4\frac{4\rho^2 x_\rho x_\sigma}{(x^2+\rho^2)^4}\,{\rm tr}\,\bar\sigma_{\mu\rho}\bar\sigma_{\mu\sigma} = \frac{96\rho^2}{g^2}\int {\rm d}^4x\,\frac{x^2}{(x^2+\rho^2)^4} = \frac{16\pi^2}{g^2} = 2S_{\rm cl}. \tag{55}$$

Finally, consider the gauge modes. We note that the gauge group element can be written as $U=\exp(\theta^a T_a)$. Then, with infinitesimal θ^a, the gauge transformation of the background field $A_\mu^{\rm cl}$ is simply given by $\partial A_\mu^{\rm cl}/\partial\theta^a = [A_\mu^{\rm cl}, T_a]$. The gauge parameter needed to keep the resulted zero modes in background gauge is

$$\Lambda_a = -\frac{\rho^2}{x^2+\rho^2}T_a \tag{56}$$

Then the gauge zero modes reads:

$$\begin{aligned} Z_\mu^{(a)} &= \frac{\partial A_\mu^{\rm cl}}{\partial\theta^a} + {\rm D}_\mu^{\rm cl}\Lambda^{(a)} = [A_\mu^{\rm cl}, T_a] + {\rm D}_\mu^{\rm cl}\left(-\frac{\rho^2}{x^2+\rho^2}T_a\right) \\ &= -\partial_\mu\Big(\frac{\rho^2}{x^2+\rho^2}\Big)T_a + \Big(1-\frac{\rho^2}{x^2+\rho^2}\Big)[A_\mu^{\rm cl}, T_a] = {\rm D}_\mu^{\rm cl}\Big(\frac{x^2}{x^2+\rho^2}T_a\Big). \end{aligned} \tag{57}$$

Or, more extensively,

$$Z_\mu^{(a)} = \frac{2\rho^2 x_\mu}{(x^2+\rho^2)^2}T_a + \frac{x^2}{x^2+\rho^2}\frac{2\rho^2\epsilon_{bac}\eta_{b\mu\nu}x_\nu}{x^2(x^2+\rho^2)}T_c \tag{58}$$

Let's check that the background gauge condition is fulfilled: $\mathrm{D}^{\mathrm{cl}}_\mu Z^{(a)}_\mu = 0$. Firstly the ∂_μ on the second term vanishes due to $\eta_{b\mu\nu}x_\mu x_\nu = 0$, The commutator between A^{cl}_μ and the first term vanishes due to $\bar{\sigma}_{\mu\nu}x_\mu x_\nu = 0$. Thus there remain two terms:

$$\mathrm{D}^{\mathrm{cl}}_\mu Z^{(a)}_\mu = \partial_\mu\left(\frac{2\rho^2 x_\mu}{(x^2+\rho^2)^2}T_a\right) + \frac{2\rho^2\epsilon_{bac}\eta_{b\mu\nu}x_\nu}{(x^2+\rho^2)^2}[A^c_\mu, T_c] = 0. \tag{59}$$

Then the norm of gauge modes can be calculated as

$$U^{ab} = \frac{4\pi^2}{g^2}\rho^2\delta_{ab} = \frac{1}{2}\delta_{ab}\rho^2 S_{\mathrm{cl}}. \tag{60}$$

It is needed to generalize this result to the entire gauge group where the gauge parameters θ^a need not to be small. In this case, we note that $U^{-1}(\theta)\frac{\partial}{\partial\theta^a}U(\theta) = e^a_\alpha(\theta)T_a$ where $e^a_\alpha(\theta)$ is the group vielbein in which α denotes curved coordinates and a denotes vielvein indices. The nontrivial group vielbein simply tells us that the group space is not flat. Now we have

$$\frac{\partial}{\partial\theta^a}A^{\mathrm{cl}}_\mu(\theta) = [A^{\mathrm{cl}}_\mu(\theta), e^a_\alpha(\theta)T_a]. \tag{61}$$

The corresponding gauge zero modes now contain the gauge parameter $\Lambda_\alpha(\theta) = -\rho^2(x^2+\rho^2)^{-1}e^a_\alpha(\theta)T_a$. Furthermore, the group metric has the form $g_{\alpha\beta} = \eta_{ab}e^a_\alpha e^b_\beta$.

It can be proved that all nondiagonal terms with different type of zero modes in the metric of moduli space vanish. Therefore now we have got the complete form of the metric U^{ij}, as:

$$U^{ij} = \begin{pmatrix} \delta^{\mu\nu}S_{\mathrm{cl}} & & \\ & 2S_{\mathrm{cl}} & \\ & & \frac{1}{2}g_{\alpha\beta}(\theta)\rho^2 S_{\mathrm{cl}} \end{pmatrix}. \tag{62}$$

This gives

$$\sqrt{\det U} = \tfrac{1}{2}S^4_{\mathrm{cl}}\rho^3\sqrt{\det g_{\alpha\beta}(\theta)} = \frac{1}{2}\cdot\Big(\frac{8\pi^2}{g^2}\Big)^4\rho^3\sqrt{\det g_{\alpha\beta}} = \frac{2^{11}\pi^8\rho^3}{g^8}\sqrt{\det g_{\alpha\beta}}. \tag{63}$$

This is the result for $SU(2)$. The result for $SU(N)$ is given by

$$\sqrt{\det U} = \frac{2N+7}{\rho^5}\Big(\frac{\pi\rho}{g}\Big)^{4N}\sqrt{\det g_{\alpha\beta}}. \tag{64}$$

We omit the derivations and refer the readers to references.

Generalization to $SU(N)$ $k=\pm1$ instantons. There arise additional $4N-5$ collective coordinates when we generalize the $k=\pm1$ $SU(2)$ instanton to $SU(N)$, besides the five zero modes associated with translation and dilatation. As studied previously, these collective coordinates correspond to the coset space $SU(N)/\big(SU(n-2)\times U(1)\big)$, the nontrivial gauge transformations on the instanton solution of embedded $SU(N)$ instanton from $SU(2)$.

4.2 Fermionic Zero Modes

In order to study fermions (i.e., in supersymmetric theories) with instantons, we also have to know the structure of their zero modes. This is the task of present subsection.

Adjoint Fermions. Firstly let us consider the fermions in the adjoint representation of $SU(2)$. As has been calculated with the aid of index theorem, there are 4 zero modes for the Dirac operator in $k=\pm 1$ instanton background. Here we claim that these four zero modes can be written as

$$\lambda^\alpha = -\tfrac{1}{2}(\sigma_{\rho\sigma})^\alpha{}_\beta\big[\xi^\beta - \sigma_\nu^{\beta\gamma'}\bar{\eta}_{\gamma'}(x-x_0)^\nu\big]F_{\rho\sigma}, \tag{65}$$

where ξ^α and $\bar{\eta}_{\gamma'}$ are four fermionic collective coordinates and $\alpha,\gamma' = 1,2$ are spin indices in Euclidean space. To check that this expression does solve the Dirac equation, we note that

$$\begin{aligned}\bar{\not{D}}_{\alpha'\beta}\lambda^\beta =& -\tfrac{1}{2}(\bar{\sigma}_\mu)_{\alpha'\beta}(\sigma_{\rho\sigma})^\beta{}_\gamma \mathrm{D}_\mu\big[\xi^\gamma - \sigma_\lambda^{\gamma\delta'}\bar{\eta}_{\delta'}(x-x_0)^\lambda\big]F_{\rho\sigma}\\ =& -\tfrac{1}{2}\big(\delta_{\mu\rho}\bar{\sigma}_\sigma - \delta_{\mu\sigma}\bar{\sigma}_\rho - \epsilon_{\mu\rho\sigma\tau}\bar{\sigma}_\tau\big)_{\alpha'\gamma}\big[\xi^\gamma - \sigma_\lambda^{\gamma\delta'}\bar{\eta}_{\delta'}(x-x_0)^\lambda\big]\mathrm{D}_\mu F_{\rho\sigma}\\ &+\tfrac{1}{2}(\bar{\sigma}_\mu\sigma_{\rho\sigma}\sigma_\lambda)_{\alpha'}{}^{\delta'}\bar{\eta}_{\delta'}(\mathrm{D}_\mu x^\lambda)F_{\rho\sigma} = 0.\end{aligned} \tag{66}$$

The last expression equals to zero because $\mathrm{D}_\rho F_{\rho\sigma} = 0$ due to the classical equations of motion, $\epsilon_{\mu\rho\sigma\tau}\mathrm{D}_\mu F_{\rho\sigma} = 0$ is the Bianchi identity and $\bar{\sigma}_\mu\sigma_{\rho\sigma}\sigma_\mu = 0$.

Now let us

Then consider the $SU(N)$'s case. Here the expression for the fermionic zero modes depend on the gauge. We consider the singular gauge only, in which the gauge potential background reads

$$A_{\mu u}{}^v = -\frac{\rho^2}{(x^2+\rho^2)}\bar{\sigma}_{\mu u}{}^v x_\nu, \tag{67}$$

where $u, v = 1,\cdots,N$ are color indices, and

$$\bar{\sigma}_{\mu u}{}^v = \begin{pmatrix} 0 & 0 \\ 0 & \bar{\sigma}_{\mu\nu\alpha}{}^\beta\end{pmatrix}$$

tells us how the $SU(2)$ is imbedded into $SU(N)$. Then the expression for λ^α still gives 4 fermionic zero modes, while now there are also additional $2(N-2)$ zero modes, given by

$$\lambda^\alpha{}_u{}^v = \frac{\rho}{\sqrt{x^2(x^2+\rho^2)^3}}(\mu_u x^{\alpha\nu} + x^\alpha{}_u\bar{\mu}^v), \tag{68}$$

where for fixed α, the N-component vectors μ_u and $x^{\alpha v}$ are given by

$$\mu_u = (\mu_1,\cdots,\mu_{N-2},0,0), \qquad x^{\alpha v} = (0,\cdots,0,x^\mu\sigma_\mu^{\alpha\beta'}) \quad \text{with} \quad N-2+\beta' = v, \tag{69}$$

and $x^\alpha{}_u = x^{\alpha u}\epsilon_{vu}$

Fundamental fermions. In this case there is only one fermionic collective coordinate as the index theorem told us. We denote this collective coordinate by $\mathcal{K}$, then the fermionic zero mode in the case of $k=-1$ $SU(N)$ singular gauge is

$$(\lambda^\alpha)_u = \frac{\rho}{\sqrt{x^2(x^2+\rho^2)^3}}x^\alpha{}_u\mathcal{K}, \tag{70}$$

4.3 The Integral Measure

Now let us move on to a study of integral measure in moduli space. We still consider the measure for bosonic collective coordinates and fermionic ones separately.

Bosonic collective coordinates. To construct the integral measure in this case, our strategy is firstly to define the integral measure with the aid of zero modes, then reexpress it in terms of collective coordinates.

To begin with, let us recall how the path integral measure is defined in the usual case for nonzero modes. Suppose the action functional is expanded in terms of quantum fluctuations to their quadratic order,

$$S = S_{\rm cl} + \tfrac{1}{2}\varphi^A M_{AB}(\phi_{\rm cl})\varphi^B, \tag{71}$$

where M_{AB} is assumed to be an Hermitian operator, and the fluctuation φ^A is given by

$$\phi^A(x) = \phi^A_{\rm cl}(x,\gamma) + \varphi^A(x,\gamma). \tag{72}$$

Then we can find a complete set of eigenfunctions F^B_α for M_{AB} by solving $M_{AB}F^B_\alpha = \epsilon_\alpha F^A_\alpha$ and expand the fluctuation field φ^A in terms of them, as

$$\varphi^A = \sum_\alpha \xi_\alpha F^A_\alpha, \tag{73}$$

with coefficients ξ_α. The norm of the eigenfunctions defines a metric on the field space, through

$$U_{\alpha\beta} = \langle F_\alpha | F_\beta\rangle = \int {\rm d}^4x\, F^A_\alpha(x) F^A_\beta(x). \tag{74}$$

Then the action becomes

$$S = S_{\rm cl} + \frac{1}{2}\sum_{\alpha,\beta} U_{\alpha\beta}\epsilon_\beta\xi_\alpha\xi_\beta, \tag{75}$$

and the integral measure is defined to be

$$[{\rm d}\phi] \equiv \prod_\alpha \sqrt{\frac{\det U_{\alpha\beta}}{2\pi}}{\rm d}\xi_\alpha. \tag{76}$$

In the same manner, we can generalize this definition to include the zero modes also. Here we have the zero modes Z^A_i instead of nonzero modes F^A_i and also the moduli metric $U_{ij} = \langle Z_i|Z_j\rangle$ instead of the metric for nonzero modes $U_{\alpha\beta}$. Then, we define the path integral measure of zero modes as

$$[{\rm d}\phi]_0 = \prod_i \sqrt{\frac{\det U_{ij}}{2\pi}}{\rm d}\xi^i_0 \tag{77}$$

Then the path integral reads

$$\int [{\rm d}\phi] e^{-S[\phi]} = \int \prod_i \sqrt{\frac{\det U_{ij}}{2\pi}}{\rm d}\xi^i_0 e^{-S_{\rm cl}}(\det{}' M_{AB})^{-1/2}. \tag{78}$$

To convert this integral measure to an integral measure on collective coordinates, we use the well-known Faddeev-Popov trick of "inserting the 1":

$$1 = \int {\rm d}\gamma_j \delta(f_i(\gamma))\left|\det\frac{\partial f_i}{\partial\gamma_j}\right|, \tag{79}$$

with $f_i(\gamma)$ chosen to be the inner product of fluctuations and the zero modes

$$f_i = -\langle\varphi|Z_i\rangle = \int {\rm d}^4x\,\varphi^A Z^A_i = \int {\rm d}^4x\,\xi^j_0 Z^A_j Z^A_i = \xi^j_0 U_{ji}. \tag{80}$$

Then we have

$$\begin{aligned} 1 &= \int \mathrm{d}\gamma_j \det\left| -\int \mathrm{d}^4x \left(\frac{\partial \varphi^A}{\partial \gamma_j} Z_i^A + \varphi^A \frac{\partial Z_i^A}{\partial \gamma_j} \right) \right| \delta(\xi_0^j U_{ji}) \\ &= \int \mathrm{d}\gamma_j \det\left| U_{ji} - \int \mathrm{d}^4x\, \varphi^A \frac{\partial Z_i^A}{\partial \gamma_j} \right| \delta(\xi_0^j U_{ji}) \\ &\xrightarrow{\text{1-loop}} \int \mathrm{d}\gamma_j \det|U_{ji}|\delta(\xi_0^j U_{ji}) = \int \mathrm{d}\gamma_j \delta(\xi_0^i). \end{aligned} \tag{81}$$

where we use the fact that the whole field $\phi^A(x)$ does not depend on γ, thus one can replace $\partial\varphi^A/\partial\gamma_j$ with $-\partial\phi_{\rm cl}^A/\partial\gamma_j = -Z_j^A$. The last expression is because the delta function insertion demands ξ_o^i to zero since U_{ij} in nonsingular; as a consequence, φ^A contains only nonzero modes, namely the true quantum fluctuation, and therefore belongs to high orders.

Then the path integral becomes

$$\int [\mathrm{d}\phi] e^{-S[\phi]} = \int \prod_i \mathrm{d}\gamma_i \sqrt{\frac{\det U_{ij}}{2\pi}} e^{-S_{\rm cl}} (\det{}' M)^{-1/2}. \tag{82}$$

Note that the integral measure in moduli space is indeed generally invariant in the sense that it keeps invariant under the general collective-coordinates transformations.

Now we can write down the integral measure for moduli space of $SU(N)$ $|k| = 1$ instanton as

$$\frac{1}{(\sqrt{2\pi})^{4N}} \left(\frac{2^{2N+7}}{\rho^5} \left(\frac{\pi\rho}{g} \right)^{4N} \sqrt{\det g_{\alpha\beta}} \right) \mathrm{d}^4x_0 \mathrm{d}\rho [\mathrm{d}\mu]_{SU(N)}, \tag{83}$$

where $[\mathrm{d}\mu]_{SU(N)}$ is the corresponding gauge group integral measure. If the correlation functions to be evaluated are gauge invariant, then the gauge group integral can be worked out, which simply gives the volume of the coset space $SU(N)/\big(SU(N-2)\times U(1)\big)$, namely

$$\mathrm{Vol}\left[\frac{SU(N)}{SU(N-2)\times U(1)} \right] = \int \sqrt{\det g_{\alpha\beta}} [\mathrm{d}\mu]_{SU(N)} = \frac{2^{4N-5}\pi^{2N-2}}{(N-1)!(N-2)!}. \tag{84}$$

Put all these factors together, we finally get the moduli measure

$$\begin{aligned} &\frac{1}{(\sqrt{2\pi})^{4N}} \left(\frac{2^{2N+7}}{\rho^5} \left(\frac{\pi\rho}{g} \right)^{4N} \right) \cdot \frac{2^{4N-5}\pi^{2N-2}}{(N-1)!(N-2)!} \mathrm{d}^4x_0 \mathrm{d}\rho \\ &= \frac{2^{4N+2}\pi^{4N-2}\rho^{4N}}{(N-1)!(N-2)!g^{4N}} \frac{\mathrm{d}\rho}{\rho^5} \mathrm{d}^4x_0. \end{aligned} \tag{85}$$

Fermionic collective coordinates. We simply present the result here.

5 One-Loop Determinants

In this section we will perform the one loop calculation of the path integral with instanton background. In particular, we will show that the cancelation of zero point energies among bosonic and fermionic contributions still happens when instantons are present.

Therefore let us consider a supersymmetric theory described by the following action

$$S = -\frac{1}{g^2} \int \mathrm{d}^4x \ \mathrm{tr} \left[\frac{1}{2} F_{\mu\nu}F_{\mu\nu} + (\mathrm{D}_\mu\phi)(\mathrm{D}_\mu\phi) - \mathrm{i}\bar\lambda\bar{\not{D}}\lambda - \mathrm{i}\lambda\not{D}\bar\lambda \right]. \tag{86}$$

We note that we are now in the Euclidean space and therefore there is no "reality condition" as the one in Minkowski spacetime that relates a Majorana spinor with its complex conjugation. That is, the two Weyl fermions λ and $\bar{\lambda}$ in the action above should be treated as being independent.

To quantize the theory, we add the gauge fixing term and the corresponding ghost term as before. Then the integration over the fluctuations of gauge potential $\mathcal{A}_\mu$ is given by

$$[\det{}'\Delta_{\mu\nu}]^{-1/2}, \qquad \text{with} \;\; \Delta_{\mu\nu} = -\mathrm{D}^2\delta_{\mu\nu} - 2F_{\mu\nu}, \tag{87}$$

where the prime on the determinant means that zero modes should be excluded. Similarly, the integration over the scalar fields gives

$$[\det \Delta_\phi]^{-1/2}, \qquad \text{with} \;\; \Delta_\phi = -\mathrm{D}^2, \tag{88}$$

and the integration over the ghost fields gives

$$[\det \Delta_{\text{gh}}], \qquad \text{with} \;\; \Delta_{\text{gh}} = -\mathrm{D}^2. \tag{89}$$

For fermions, it's nonsense to talk about the eigenvalue problem of $\not{\mathcal{D}}$ or $\bar{\not{\mathcal{D}}}$ since these two operators map the functions in one space to the other. The way out of this trouble is familiar; we consider instead the "squared" operators

$$\Delta_- \equiv -\not{\mathcal{D}}\bar{\not{\mathcal{D}}} = -\mathrm{D}^2 - \tfrac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}, \qquad\qquad \Delta_+ \equiv -\bar{\not{\mathcal{D}}}\not{\mathcal{D}} = -\mathrm{D}^2, \tag{90}$$

and define the fermionic path integral measure through eigenfunction expansion of these two operators. Then the integration can be finished to give the result $[\det{}'\Delta_-]^{1/4}[\det\Delta_+]^{1/4}$. Note that Δ_- has zero modes and Δ_+ doesn't.

We all know that integrating out these fluctuations from their quadratic action gives rises to vacuum energy at 1-loop level. Since in a supersymmetric theory one expects that these contributions to vacuum energy sum to zero, so let us now check the cancelation among the determinants we have got. Firstly we note that $\det\Delta_\phi = \det\Delta_{\text{gh}} = [\det\Delta_+]^{1/2}$. For vector field determinant $\det\Delta_{\mu\nu}$ one can also show that $\det{}'\Delta_{\mu\nu} = [\det{}'\Delta_-]^2$.

Now let us consider the one-loop determinant for a Yang-Mills system (including ghosts) coupled to n real adjoint scalars and $\mathcal{N}$ adjoint Weyl spinors. The total one-loop determinant is

$$[\det{}'\Delta_-]^{-1+\mathcal{N}/4}[\det\Delta_+]^{(2+\mathcal{N}-n)/4}. \tag{91}$$

In the particular case that $\mathcal{N} - \frac{1}{2}n = 1$, the total determinant becomes a power of $\det{}'\Delta_-/\det\Delta_+$. We note that this condition is obviously fulfilled by a SUSY theory. Therefore we conclude that all nonzero modes contribute a zero vacuum energy provided that $\det\Delta_- = \det\Delta_+$, while the latter equality is true after regularization only when the regulation procedure respects supersymmetry.

5.1 The Exact β Function for SYM Theories

To calculate the zero mode contributions to the beta function in SYM theories, we adopt the viewpoint that the regularized partition function corresponds to the vacuum energy, which is an observable and should not depends on the regularization parameter. This requirement gives a constraint on the dependence of the bare coupling constant g on the regularization scale M. It can be shown that the beta function obtained in this way is equivalent to the usual definition in which the beta function reflects the running of the renormalized coupling with the renormalization scale, provided that the theory is renormalizable.

The zero mode contribution comes from the integration over the moduli space. Therefore we firstly write down the corresponding integral measure as follows:

$$\mathrm{d}\mathcal{M} = e^{-8\pi^2/g^2}\left[\frac{2^{4N+2}\pi^{4N-2}}{(N-1)!(N-2)!}\Big(\frac{\rho}{g}\Big)^{4N}\Big(\frac{M}{\sqrt{2\pi}}\Big)^{4N}\frac{di^4x\mathrm{d}\rho}{\rho^5}\right] \\ \times\left[\frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{4S_{\rm cl}M}\frac{\mathrm{d}\bar\eta_1\mathrm{d}\bar\eta_2}{8\rho^2S_{\rm cl}M}\frac{1}{\big(\frac{1}{4}S_{\rm cl}M\big)^{N-2}}\prod_{u=1}^{N-2}\mathrm{d}\mu^u\mathrm{d}\bar\mu_u\right] \tag{92}$$

Let us extract the g and M dependence from this measure:

$$\mathrm{d}\mathcal{M}\propto e^{-8\pi^2/g^2}M^{3N}\Big(\frac{1}{g}\Big)^{2N}. \tag{93}$$

Requiring this measure to be independent of M yields,

$$M\frac{\partial}{\partial M}\bigg(-\frac{8\pi^2}{g^2}+3N\log M-2N\log g\bigg), \tag{94}$$

thus

$$\beta\equiv M\frac{\partial}{\partial M}g=\frac{3N}{\frac{2N}{g}-\frac{16\pi^2}{g^3}}. \tag{95}$$

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