* Instantons as most probable tunnelling path.

— In the third talk on instantons we're going to study the quantum tunneling between vacua in Minkowski space with different winding numbers. We will show that the most probable path of this tunneling process is given by a field configuration that corresponds to the instanton solution in Euclidean space.

<1>

Firstly let us clarify how the winding number Q is defined in Minkowski spacetime. Recall that the Euclidean winding number k is defined to be the integration of $\epsilon^{\mu\nu\rho\sigma} tr(F_{\mu\nu} F_{\rho\sigma})$ over the whole space, with a properly chosen normalization. In Minkowski spacetime, a similar definition still applies, for, the integrand $\epsilon^{\mu\nu\rho\sigma} tr F_{\mu\nu} F_{\rho\sigma}$ is independent of the choice of metric. (as we have shown in the first talk when proving the vanishing of $T_{\mu\nu}$)

The only difference here is the region of the integration. (and the normalization, probably). That is, we define Q by:

$$Q = -\frac{1}{64\pi^2}\int_{\sigma_1}^{\sigma_2} d^4x\, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$$

$$= -\frac{1}{4\pi^2}\int_{\sigma_1}^{\sigma_2} d^4x\, \mathrm{tr}(\vec{E}\cdot\vec{B}).$$

where σ_1 and σ_2 are two 3d hypersurfaces at time $t=t_1$ and $t=t_2$, respectively $(t_1 < t_2)$. The "electric" and "magnetic" field strength are defined in the usual way, namely, $E_i = -F_{0i}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. The quotation marks indicate that we're dealing with non-abelian gauge field with $SU(2)$ group. (We'll restrict ourselves to the case of $SU(2)$ in this note, although the generalization to $SU(N)$ is quite straightforward)

(2).

The problem we're going to study is the tunnelling between two vacua, located in σ_1 at t_1 and σ_2 at t_2.

On σ_1 and σ_2, the gauge potential $A_\mu(\vec{x}, t)$ stays in the vacuum configuration, with vanishing energy, namely $H = \frac{1}{2}\left[(\vec{E}^a)^2 + (\vec{B}^a)^2\right] = 0 \Rightarrow$

$\vec{E}^a = \vec{B}^a = 0 \Rightarrow F^a_{\mu\nu} = 0 \Rightarrow A_\mu(\vec{x}, t)$ is a pure gauge on σ_1 and σ_2. That is, we can write:

$$\begin{cases} A_\mu(\vec{x}, t_1) = e^{-\alpha(\vec{x}, t_1)} \partial_\mu e^{\alpha(\vec{x}, t_1)} \\ A_\mu(\vec{x}, t_2) = e^{-\alpha(\vec{x}, t_2)} \partial_\mu e^{\alpha(\vec{x}, t_2)} \end{cases}$$

To simplify the problem, we choose the temporal gauge condition $A_0(\vec{x}, t) = 0$ for all $\vec{x}$ and t. Since the tunnelling rate, as a physical quantity, should be independent of the gauge choice as will been shown in the following.

There're still some gauge redundancy even when we fix the gauge by $A_0 = 0$, a phenomanon we're all familiar with. Actually, this gauge redundancy consists of all gauge transformations with time independent gauge parameter, namely,

$$A_0'(\vec{x}, t) = e^{-g(\vec{x})} \partial_0 e^{g(\vec{x})} = 0.$$

We can therefore make use of this DOF to further restrict $\alpha(\vec{x}, t_1) = 0$, such that the gauge potential on σ_1 is not only a pure gauge, but also vanishes entirely:

$$A_\mu(\vec{x}, t_1) = 0, \quad \text{on } \sigma_1.$$

The tunnelling we're considering should be such that the energy of the configuration $A_\mu(\vec{x}, t)$ be finite for all $t \in [t_1, t_2]$, otherwise there will be an infinitely high energy barrier that makes the quantum tunnelling impossible. Therefore we must have

$$T_{\mu\nu} \to 0 \quad \text{as} \quad |\vec{x}| \to \infty$$

$$\Rightarrow \quad F_{\mu\nu} \to 0 \quad \text{as} \quad |\vec{x}| \to \infty$$

$$\Rightarrow \quad A_\mu(\vec{x}, t) \to e^{-\alpha(\vec{x}, t)} \partial_\mu e^{\alpha(\vec{x}, t)}, \ \text{as} \ |\vec{x}| \to \infty.$$

Furthermore, the gauge condition $A_0 \equiv 0$ gives $\partial_0 e^{\alpha(\vec{x}, t)} = 0 \Rightarrow \alpha(\vec{x}, t)$ does not depend on time. as $|\vec{x}| \to \infty$.

In addition, we know that $\alpha(\vec{x}, t_1) = 0$

$$\Rightarrow \quad \alpha(\vec{x}, t) = 0 \ \text{for} \ \forall t \ \text{and} \ |\vec{x}| \to \infty.$$

In particular,

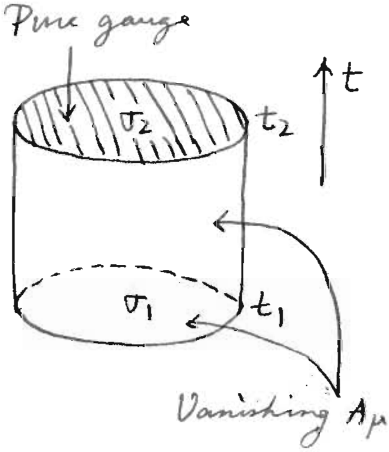
$$\alpha(\vec{x}, t_2) = 0 \ \text{at} \ |\vec{x}| = \infty$$

$$\Rightarrow \ A_i(\vec{x}, t_2) = 0 \ \text{at} \ |\vec{x}| = \infty.$$

The vanishing of $A_i(\vec{x}, t)$ at $|\vec{x}| = \infty$ for all t allows us to compactify the 3d hypersurfaces to spheres S_3, with the north pole identified with $|\vec{x}| = \infty$ where the gauge potential vanishes. With this picture, we see that the Minkowski winding number Q defined above now becomes

$$Q = \frac{1}{8\pi^2}\int d^4x \, \epsilon^{\mu\nu\rho\sigma} \, tr\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma\right)$$

$$\xrightarrow[\text{boundary}]{F_{\mu\nu}=0 \text{ on the}} -\frac{1}{24\pi^2}\int d\sigma_\mu \, \epsilon^{\mu\nu\rho\sigma} \, tr\, A_\nu A_\rho A_\sigma$$



Now, only the top of the cylinder (σ_2) contributes to this integral (as shown in the figure). thus we see that Q indeed characterizes a map from S_3 to $SU(2)$, and describes the winding of the gauge field on S_3.

<3>.

Now let us begin the main topic, the tunnelling. The first claim we make here is that the path for the tunnelling process cannot be the solution of the equations of motion, if the initial and final vacua have different winding number. (By the tunnelling path we mean a field configuration $A_\mu(\vec{x},t)$ lying in Minkowski space between the two hypersurfaces σ_1 and σ_2 such that $A_\mu(\vec{x},t_1)$ and $A_\mu(\vec{x},t_2)$ are precisely the two vacua we're considering.)

The reason for this claim is that $F_{\mu\nu}=0$ at t_1, which implies that $F_{\mu\nu}=0$ everywhere, for any $\vec{x}$ and t. [We leave the proof to the end.] Then the winding number $Q \sim \int d^4x\, E\cdot B$ will vanish identically.

With this in mind, let's consider a given path connecting the two vacua, $A_j(\vec{x},t)$. ~~and~~ together with a class of paths $A_j^{(\lambda)}(\vec{x},t)$ as deformations of $A_j(\vec{x},t)$. The deformation is given by

$$A_j^{(\lambda)}(\vec{x},t) = A_j(\vec{x},\lambda(t))$$

with $\lambda(t)$ a smooth function satisfying $\lambda(t_1)=t_1$ and $\lambda(t_2)=t_2$. The task for us is to find the path that maximize the tunnelling rate. Then for the given path $A_j(\vec{x},t)$, this task becomes a problem of variations on the deformation function $\lambda(t)$. [One may immediately ask what if we choose a different path $A_j'(\vec{x},t)$ to begin with? We'll come back to this question shortly.]

To formulate this variation problem, let us evaluate the classical action explicitly with the class of deformations $\lambda(t)$. Firstly, we calculate the "electric" and "magnetic" field strength:

$$E_j = -F_{0j} = -\partial_0 A_j^{(\lambda)}(\vec{x},t) + \partial_j \underbrace{A_0^{(\lambda)}(\vec{x},t)}_{=\,0 \text{ by the gauge condition.}} - [A_0, A_j]$$
$$= -\frac{\partial A_j}{\partial \lambda}(\vec{x}, \lambda(t))\,\dot\lambda\,;$$

$$B_i = \tfrac{1}{2}\epsilon_{ijk}F_{jk} = \tfrac{1}{2}\epsilon_{ijk}\Big[\partial_j A_k(\vec{x},\lambda(t)) + A_j(\vec{x},\lambda(t))A_k(\vec{x},\lambda(t)) - (j\leftrightarrow k)\Big]$$

Then the Lagrangian (not density) is,

$$L = \int d^3x \,\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu})^2 = -\frac{1}{g^2}\int d^3x\,\mathrm{tr}(\vec{E}^2 - \vec{B}^2)$$

which can be written as,

$$L = \tfrac{1}{2}m(\lambda)\dot\lambda^2 - V(\lambda),$$

with

$$m(\lambda) = -\frac{2}{g^2}\int d^3x\,\mathrm{tr}\left(\frac{\partial\vec{A}}{\partial\lambda}\right)^2 \geq 0$$

$$V(\lambda) = -\frac{1}{g^2}\int d^3x\,\mathrm{tr}\,\vec{B}^2 \geq 0.$$

The picture can be viewed as a system of one particle mechanics with position-dependent mass with a potential $V(\lambda)$, in which λ is the canonical momentum. Then the Hamiltonian is

$$H = \frac{p^2(\lambda)}{2m(\lambda)} + V(\lambda)$$

with the canonical momentum $p(\lambda) = \frac{\partial}{\partial\dot\lambda}L = m(\lambda)\dot\lambda$.

But the tunnelling rate for this system is easy to obtained. By quantum mechanics, this is given by e^{-2R}, with

$$R = \int_{\lambda_1}^{\lambda_2} d\lambda\,\sqrt{2m(\lambda)(V(\lambda)-E)}\,,$$

with the boundary conditions on σ_1 and σ_2

specified by $V(\lambda) = m(\lambda) = 0$ at λ_1 and λ_2. ($\lambda_1 \equiv \lambda(t_1) = t_1$; $\lambda_2 \equiv \lambda(t_2) = t_2$). Furthermore, one can set $E = 0$.

The trick of finding the maximum of this tunnelling rate is the triangle inequality. To see how this works, let us rewrite the rate e^{-2R} as

$$R = \int_{\lambda_1}^{\lambda_2} d\lambda \, 2 \left[\frac{1}{g^2} \int d^3x \, \mathrm{tr} \left(\frac{\partial \vec{A}}{\partial \lambda} \right)^2 \right]^{1/2} \left[\frac{1}{g^2} \int d^3x \, \mathrm{tr} \, \vec{B} \right]^{1/2}$$

$$= \frac{2}{g^2} \int_{t_1}^{t_2} dt \left[\int d^3x \, \mathrm{tr} \, \vec{E}^2 \right]^{1/2} \left[\int d^3x \, \mathrm{tr} \, \vec{B}^2 \right]^{1/2}$$

Now, the triangle inequality tells us that

$$R \geq \frac{1}{g^2} \left| \int_{t_1}^{t_2} d^4x \, \mathrm{tr} \, (\vec{E} \cdot \vec{B}) \right| = \frac{8\pi^2}{g^2} |Q|$$

$$\Rightarrow \quad e^{-R} \leq e^{-8\pi^2 |Q| / g^2}.$$

The equality holds when $\vec{E} \parallel \vec{B}$, and gives our desired result. We note that this result is gauge-independent. Furthermore, such a configuration corresponds to the Euclidean instanton.

⟨4⟩. Application: the Strong CP problem.

As we discussed previously, the gauge vacua with different winding numbers can tunnel to each other. Therefore one may expect that the physical vacuum should be a superposition of all vacua with all possible winding numbers. As a consequence, one expects that the vacuum should be invariant under a "gauge" transformation that changes the winding number. That is, if we define a transformation T by $T|n\rangle = |n+1\rangle$, then the vacuum state $|vac\rangle$ should be such that $T|vac\rangle = e^{i\varphi}|vac\rangle$. That is, the vacuum is an eigenstate of the "raising operator" T. We all know that the eigenstates of such raising operators are "coherent states". Therefore we have

$$|vac\rangle \equiv |\theta\rangle = \sum_n e^{in\theta}|n\rangle.$$

$$\Rightarrow \; T|\theta\rangle = \sum e^{in\theta}|n+1\rangle = e^{-i\theta}|\theta\rangle.$$

— Where does the θ-angle come from?

⟨a⟩. The most direct source of this θ-angle is the θ-term in QCD:

$$\mathcal{L}_\theta = -\theta_{QCD}\frac{g^2}{16\pi^2}\,\mathrm{tr}\,F_{\mu\nu}{}^*F_{\mu\nu},$$

which contributes a finite factor $e^{in\theta}$ into the path integral.

(b). Another source is the electroweak sector, where the nontrivial ~~nonzero CP phase in the~~ CKM matrix can ~~be~~ also contribute an effective θ.

Recall that the SM Lagrangian for Yukawa couplings is given by:

$$\mathcal{L} = \left[- y^u_{mn} \bar{Q}_{Lm} (i\tau_2 H^*) q^u_{Rn} - y^d_{mn} \bar{Q}_{Lm} H q^d_{Rn} \right] + h.c.$$

The point is, a chiral transformation is needed in order to bring quarks to their mass eigenbase, namely, to diagonalize y_{mn}. But this chiral transformation will in turn generate a θ term from the path-integral measure (chiral anomaly!).

$$\mathcal{L}_\theta = -(\theta_{QCD} + \theta_{EW}) \frac{g^2}{32\pi^2} F^a_{\mu\nu} {}^*F^a_{\mu\nu}.$$