Note on Cosmology

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1 Introduction

2 The Homogeneous and Isotropic Universe

Our universe appears to be a system with 3 spatial dimensions and 1 temporal dimension.

2.1 Weighing the universe

Before going into a more detailed study of the universe, we firstly have a brief look at the constituents of the universe from the observational side. In the homogeneous and isotropic case, this amounts to figuring out the main contents of the universe, and measuring their energy densities. According to our understanding today, our universe consists of matter, radiation, and "dark energy". This bold classification is from the viewpoint of the

2.1.1 Photon

The energy density of photon can be inferred from its temperature. This is because photons, although decoupled long ago from the rest of the world, still remain thermally distributed (as measured beautifully by COBE satellite). So its energy density is given by,

$$\rho_\gamma = \frac{\pi^2}{15} T^4, \tag{1}$$

where $T = 2.7\text{K}$ is the well-known temperature of CMB observed today.

2.1.2 Baryonic matter

The baryonic matter here includes all atoms, neutral and ionized, as well as electrons, a bold nomenclature from cosmology community.

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2.2 Equations of motion for the homogeneous and isotropic universe

The universe is a coupled system of energy and spacetime. The dynamics of spacetime is governed by the Einstein's equation from general relativity, and the dynamics of matter distribution is governed by Boltzmann's equation. At homogeneous and isotropic level, there is only one dynamical variable for spacetime, namely the scale factor $a = a(t)$ as a function of time t, while the dynamical variables for matter contents are simply given by the energy density $\rho_i(t)$, or equivalently, the number density $n_i(t)$. In this case, the Einstein's equation reduces the Friedmann's equation,

$$\left(\frac{\mathrm{d}a/\mathrm{d}t}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i\,, \tag{2}$$

while the Boltzmann's equation reduces to a set of differential equations for variables $n_i(t)$. Now we derive them.

The Boltzmann's equation, roughly speaking, is the statement that the rate of change for the number of particles in a given phase space element, equals to the rate of collisions kicking the particle into the given phase space element, minus the rate of collision kicking the particle out of the phase space element. For the 2 to 2 process of $1+2 \to 3+4$, the only case we are interested in, this statement can be represented by the following equation for species 1,

$$\begin{aligned}\frac{1}{a^3}\frac{\mathrm{d}(n_1a^3)}{\mathrm{d}t} =& \int\prod_{i=1}^4\frac{\mathrm{d}^3p_i}{(2\pi)^3 2E_i}\,|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\\ &\times\Big[f_3f_4(1\pm f_1)(f\pm f_2)-f_1f_2(1\pm f_3)(1\pm f_4)\Big],\end{aligned} \tag{3}$$

where $\mathcal{M}$ is the amplitude for the process, $f_i = f_i(x,p,t)$ is the distribution function of species $i = 1,2,3,4$. When all species remain in thermal equilibrium, and when quantum effects can be neglected, the distribution function takes the familiar form in classical statistics, $f(E) \propto e^{-(E-\mu)/T}$, with T the temperature and μ the chemical potential. Then, the second line of the Boltzmann's equation (3) reduces to $e^{-(E_1+E_2)/T}\big(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T}\big)$, and the number density n_i is given by,

$$n_i = g_i e^{\mu_i/T}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,e^{-E_i/T}. \tag{4}$$

When a given species of mass m_i is in thermal equilibrium with free energy and particle exchange, its chemical potential is zero, and its number density, denoted as $n_i^{(0)}$, can be integrated out from the above expression to be,

$$n_i^{(0)} = g_i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,e^{-E_i/T} = \begin{cases} g_i\Big(\dfrac{m_iT}{2\pi}\Big)^{3/2}e^{-m_i/T}, & m_i \gg T,\\ g_i\,\dfrac{T^3}{\pi^2}, & m_i \ll T.\end{cases} \tag{5}$$

Then we can reexpress chemical potential in terms of number density, through, $e^{\mu_i/T} = n_i/n_i^{(0)}$.

Then the second line of (3) further reduces to,

$$e^{-(E_1+E_2)/T}\left(\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}}-\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}\right)$$

On the other hand, we define the thermally averaged cross section $\langle\sigma v\rangle$ to be,

$$\langle\sigma v\rangle\equiv\frac{1}{n_1^{(0)}n_2^{(0)}}\int\prod_{i=1}^{4}\frac{\mathrm{d}^3p_i}{(2\pi)^32E_i}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)e^{-(E_1+E_2)/T},\tag{6}$$

then the Boltzmann's equation (3) simplifies to,

$$\frac{1}{a^3}\frac{\mathrm{d}(n_1a^3)}{\mathrm{d}t}=n_1^{(0)}n_2^{(0)}\langle\sigma v\rangle\left(\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}}-\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}\right).\tag{7}$$

When some components deviate from thermal equilibrium, namely when the rate of collision is comparable with the rate of cosmic expansion, this is the proper equation we will work with. However, when the collision rate $n_2\langle\sigma n\rangle$ is much larger than the expansion rate H, the left hand side of above equation can be neglected, then we get the following equation,

$$\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}}=\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}},\tag{8}$$

which is known as Saha equation.

In the following several subsections we apply the equations derived above to three processes in early universe: the decoupling of dark matter, the big-bang nucleosynthesis, and the recombination.

2.3 Big-bang nucleosynthesis

When the temperature of the universe is around 1MeV, we have:

- Relativistic particles in thermal contact: photon γ and electron/positron $e^{\pm}$. They have the same abundance, ignoring the tiny difference between Bose-Einstein and Fermi-Dirac statistics (2 helicity degrees for each of $\gamma,\ e^-,\ e^+$).
- Decoupled component: relativistic neutrino. Although decoupled, neutrinos have the same temperature as photons and electrons, thus the same abundance for each helicity degree.
- Nonrelativistic particles: hadron. No anti-hadrons around 1MeV. The abundance of hadrons is usually represented in the ratio $\eta_b\equiv n_b/n_\gamma=5.5\times10^{-10}(\Omega_bh^2/0.020)$.

The task then is to determine the fate of hadrons, namely the formation of chemical elements. To solve the problem, we introduce two simplifications:

1. Practically only isotopes of hydrogen and helium can be formed significantly, besides there is also trace amount of lithium, around 10^{-9}. This is due to the large binding energy of ^{4}He.

2. When $T \gtrsim 0.1$MeV, there are only free protons and neutrons, and no nuclei exist with significant amount. This is because, nuclei, once formed, are soon destroyed by high energy photons since the latter are very abundant.

To understand the second point above, we make a simple estimate, by considering the process $n + p \to D + \gamma$. Photons are in thermal equilibrium so $n_\gamma = n_\gamma^{(0)}$, and,

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}} = \frac{3}{2\times 2}\Big(\frac{2\pi m_D}{m_n m_p T}\Big)^{3/2} e^{B_D/T} \simeq \frac{3}{4}\Big(\frac{4\pi}{m_p T}\Big)^{3/2} e^{B_D/T},$$

where $B_D = m_n + m_p - m_D = 2.22$MeV is the binding energy of D. Now that $n_n, n_p \sim n_b$, we have,

$$\frac{n_D}{n_b} \sim \frac{n_b}{n_\gamma}\cdot n_\gamma^{(0)} \cdot \Big(\frac{4\pi}{m_p T}\Big)^{3/2} e^{B_D/T} \sim \eta_b \Big(\frac{T}{m_p}\Big)^{3/2} e^{B_D/T}, \tag{9}$$

where we used $n_\gamma = n_\gamma^{(0)}$ and $n_\gamma^{(0)} = 2T^3/\pi^2$. So when B_D/T is not too large, $n_D/n_b \ll 1$ due to $\eta_b \ll 1$. That is, no light nuclei were formed when $T \gtrsim 0.1$MeV.

So we first consider the abundance of neutron during this period of time.

2.3.1 Abundance of neutron

Our problem is to compute the ratio n_n/n_p. Protons and neutrons are in thermal equilibrium through weak process $n + \nu_e \leftrightarrow p + e^-$ and $n + e^+ \leftrightarrow p + \bar{\nu}_e$. Then, using $E = m + p^2/2m$, we have,

$$\frac{n_p^{(0)}}{n_n^{(0)}} = \frac{e^{-m_p/T}\int \mathrm{d}p\, p^2 e^{-p^2/2m_p T}}{e^{-m_n/T}\int \mathrm{d}p\, p^2 e^{-p^2/2m_n T}} \simeq e^{Q/T},$$

with $Q \equiv m_n - m_p = 1.293$MeV. Then, $n_p^{(0)} = n_n^{(0)}$ when $T \gg Q$, and $n_n^{(0)}$ goes down as $T \lesssim Q$, and would go to zero if weak interaction is efficient enough, which is actually not. So $n_n^{(0)}$ will go down toward a nonzero constant when the expansion gets faster than the collision above, if neutron would not decay. Now let us determine the fraction of neutron as the universe expands.

For clarity we define $X_n \equiv n_n/(n_n + n_p)$ to be the neutron fraction. Its time dependence can be read from the Boltzmann equation of n_n, with process $n + \nu_e \to p + e^-$. Recall that light particles ν_e and e^- are in thermal equilibrium, thus $n_\nu = n_e = n_\ell^{(0)}$. Then, Boltzmann's equation (7) becomes,

$$\frac{1}{a^3}\frac{\mathrm{d}(n_n a^3)}{\mathrm{d}t} = \lambda_{np}\bigg(\frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n\bigg), \tag{10}$$

where the collision rate $\lambda_{np} \equiv n_\ell^{(0)}\langle\sigma v\rangle$. Now,

$$a^{-3}\frac{\mathrm{d}(n_n a^3)}{\mathrm{d}t} = a^{-3}\frac{\mathrm{d}((n_n+n_p)X_n a^3)}{\mathrm{d}t} = \underbrace{(n_n+n_p)a^3}_{\text{conserved}} \cdot a^{-3}\frac{\mathrm{d}X_n}{\mathrm{d}t},$$

therefore we have,

$$\frac{\mathrm{d}X_n}{\mathrm{d}t} = \lambda_{np}\Big((1-X_n)e^{-Q/T} - X_n\Big). \tag{11}$$

To simplify this equation, we change the variable $T \to x \equiv Q/T$, then, $\frac{\mathrm{d}X_n}{\mathrm{d}t} = \frac{\mathrm{d}X_n}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$, and,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{T}\frac{\mathrm{d}T}{\mathrm{d}t}, \quad T \propto 1/a \;\Rightarrow\; \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = -H = -\sqrt{\frac{8\pi G}{3}\rho} \text{ (Friedmann's equation).}$$

The energy density, during this era of radiation domination, is given by,

$$\rho = \frac{\pi^2}{30}T^4\bigg(\sum_{i\in\text{bosons}} g_i + \frac{7}{8}\sum_{i\in\text{fermions}} g_i\bigg) \equiv g_*\frac{\pi^2}{30}T^4,$$

in which only relativistic degrees are counted. When $T \simeq 1\text{MeV}$, we have $g_* \simeq 10.75$, which is contributed from 2 degs. from photon, 6 from neutrinos (3 generations with their antiparticles) and 4 from $e^\pm$. So,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xH = x\Big(\frac{8\pi G}{3}\cdot g_*\frac{\pi^2}{30}\Big)^{1/2}\frac{Q^2}{x^2} = \frac{H(x=1)}{x},$$

where $H(x=1) = \sqrt{10.75\times 4\pi^3 GQ^4/45} \simeq 1.13\text{sec}^{-1}$. Then,

$$\frac{\mathrm{d}X_n}{\mathrm{d}x} = \frac{x\lambda_{np}}{H(x=1)}\Big[e^{-x} - X_n(1+e^{-x})\Big]. \tag{12}$$

Now let us calculate the collision rate λ_{np} for the process $n + \nu_e \to p + e^-$, given by,

$$\begin{aligned}
\lambda_{np} &= \frac{1}{n_n^{(0)}}\int \frac{\mathrm{d}^3 p_n}{(2\pi)^3 2m_n}\frac{\mathrm{d}^3 p_p}{(2\pi)^3 2m_p}\frac{\mathrm{d}^3 p_e}{(2\pi)^3 2p_e}\frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2p_\nu} \\
&\quad\times |\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_n + p_\nu - p_p - p_e)e^{-(m_n+p_\nu)/T} \\
&= \frac{1}{4m_n}\int \frac{\mathrm{d}^3 p_p}{(2\pi)^3 2m_p}\frac{\mathrm{d}^3 p_e}{(2\pi)^3 2p_e}\frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2p_\nu}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_n + p_\nu - p_p - p_e)e^{-p_\nu/T} \\
&= \frac{1}{4m_n}\cdot\frac{1}{2m_p}\int \frac{\mathrm{d}^3 p_e}{(2\pi)^3 2p_e}\frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2p_\nu}|\mathcal{M}|^2(2\pi)\delta(Q + p_\nu - p_e)e^{-p_\nu/T},
\end{aligned}$$

where we have take n and p to be nonrelativistic, e and ν_e to be ultra-relativistic. The squared amplitude, which can be read from the 4-fermion interaction, is $|\mathcal{M}|^2 = 32G_F^2(1+3g_A^2)m_p^2 p_e p_\nu$. The axial-vector coupling of neutron g_A can be obtained by measuring the lifetime of neutron $\tau_n = 886.7\text{sec}$, which can be calculated, also with 4-fermion interaction, to be $\tau_n^{-1} = \lambda_0 G_F^2(1+3g_A^2)m_e^5/2\pi^3$, where

$$\lambda_0 = \int_1^{Q/m_e}\mathrm{d}\xi\,\xi(\xi - Q/m_e)^2(\xi^2-1)^{1/2} = 1.636.$$

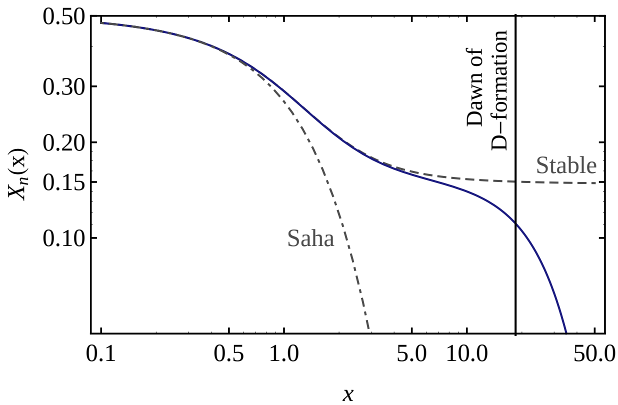
Then we can continue evaluating λ_{np},

$$\lambda_{np} = 2\times\frac{G_F^2(1+3g_A^2)m_p}{(2\pi)^5 m_n}\int \mathrm{d}^3 p_e \mathrm{d}^3 p_\nu\,\delta(Q + p_\nu - p_e)e^{-p_\nu/T} \simeq \frac{255}{\tau_n x^5}(12 + 6x + x^2).$$

Here we multiply the collision rate by 2 to take account of the two processes of the same rate.

Now we are ready to solve the differential equation (12) numerically. This solution works well for $T \gtrsim 1$MeV. Below this temperature, two more effects should be taken into account: 1) Neutron decay $n \to p + e^- + \bar{\nu}_e$; 2) Deuterium formation $n + p \to D + \gamma$. The first effect can be accommodated conveniently by including a decay factor e^{-t/τ_n}, using the $t - T$ relation $t = 132\text{sec}(0.1\text{MeV}/T)^2$. To estimate the starting of efficient Deuterium formation, we go back to (9). Then the starting point $T = T_{\text{nuc.}}$ of D-formation should be such that $n_D/n_b \sim \mathcal{O}(1)$, then $\log \eta_b + \frac{3}{2}\log(T_{\text{nuc.}}/m_p) \sim -B_D/T_{\text{nuc.}} \;\Rightarrow\; T_{\text{nuc.}} \simeq 0.07$MeV. The neutron abundance around this time is $X_n \exp(-\frac{132}{886.7}(\frac{0.1}{0.07})^2) = 0.11$.

The above results can be summarized in the following figure. The dot-dashed Saha curve shows the estimate of (8), namely the estimate with thermal equilibrium. It works well above $T > Q$. The deviation from equilibrium becomes important for $T \lesssim Q$. The dashed "stable" line shows the solution of (12) without neutron decay. The solution together with neutron decay factor is shown by solid line.



In fact, nearly all neutrons go into ^{4}He because it has a much larger binding energy than D. So we get the final abundance for ^{4}He, represented by $X_4 \equiv 4n(^4\text{He})/n_b$, to be $X_4 = 2X_n(T_{\text{nuc.}}) \simeq 0.22$. More accurate solution taking account of (quantum statistics + finite electron mass + time variation of g_*) gives,

$$X_4 = 0.2262 + 0.0135\log(\eta_b/10^{-10}).$$

A feature of this solution is that X_4 depends on η_b only logarithmically. This dependence can be understood in this way: (9) tells us roughly that $\log \eta_b \propto T_{\text{nuc.}}^{-1}$. On the other hand, $X_4 = 2X_n(T_{\text{nuc.}}) \propto e^{-t_{\text{nuc.}}/\tau_n}$. Then one may naively conclude from these two expressions that $X_4 \propto \eta_b$ rather than $\log \eta_b$. However, the crucial fact here is that D-formation begins right after neutrons start decaying, so the exponential $X \propto e^{-t/\tau_n}$ can be well approximated by a linear function in this region. As a result, we have $X_4 \sim \log \eta_b$.

There are still some deuterium formed, since the process $\text{D} + p \to {}^3\text{He} + \gamma$ is not completely efficient. More accurate computation reveals that D is formed right after $T = T_{\text{nuc}}$, and the number density finally reaches $10^{-5} \sim 10^{-4}$. The important thing here is that n_D depends on η_b rather sensitively, and this sensitive dependence can be used to infer the baryon density through the measured n_D. n_D can be measured quite accurately by looking at absorbing spectrum of gases at $z \sim 3$, which gives $D/H \sim 3 \times 10^{-5}$, and $\Omega_b h^2 \simeq 0.02$.

2.4 Recombination

Although H has the binding energy 13.6eV, the abundance of neutral H is very low around $T \sim 1$eV, again due to small η_b. Now we use Saha approximation to estimate the temperature of recombination. When $e^- + p \to \mathrm{H} + \gamma$ is in equilibrium, $n_e n_p / n_H = n_e^{(0)} n_p^{(0)} / n_H^{(0)}$. (Remember that $n_\gamma = n_\gamma^{(0)}$.) The electric neutrality requires $n_e = n_p$. So let's define $X_e \equiv n_e/(n_e + n_H) = n_p/(n_p + n_H)$. Then the above Saha equation becomes,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\Big(\frac{m_e T}{2\pi}\Big)^{3/2} e^{-(m_e + m_p - m_H)/T}.$$

As a rough estimation, ignoring ^{4}He, $n_e + n_H \simeq n_b = \eta_b n_\gamma \sim 10^{-9} T^3$. When $T \sim 13.6$eV, the right hand side of above expression is roughly $10^9 (m_e/T)^{3/2} \simeq 10^{15} \;\Rightarrow\; X_e \simeq 1$. As T is much lower than 13.6eV, X_e goes down, and we need to solve Boltzmann equation (7).

2.5 Dark matter decoupling

3 Theory of Cosmological Perturbations

When studying the fluctuations on the homogenous and isotropic background, we need to perturb both Boltzmann's equation and Einstein's equation. In this section we derive the first-order perturbation to these equations.

3.1 Boltzmann's equations of first-order perturbations

In last section we used the integrated Boltzmann's equation, where the left hand side is the time derivative of number density. In order to study the distribution more carefully, including the variation in space and momentum, we remove the integration over the particle 1's phase space, which leads to a more general equation, $\mathrm{d}f(x, P, t)/\mathrm{d}t = C[f]$, where $C[f]$ is the collision integral. In the following we derive the 1st-order perturbed form of this equation for several components of early universe.

3.1.1 Photon: collisionless part

Let's begin with the photon. The left hand side of the equation is,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial f}{\partial x^i} + \frac{\mathrm{d}P^\mu}{\mathrm{d}t}\frac{\partial f}{\partial P^\mu}, \tag{13}$$

where the 4-momentum $P^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$, and λ is the affine parameter of photon's world-line. The coefficients $\mathrm{d}x^i/\mathrm{d}t$ and $\mathrm{d}P^\mu/\mathrm{d}t$ can be obtained from photon's equation of motion, namely its geodesic equation.

We will consider scalar perturbation at the moment and we choose the conformal Newtonian gauge, so that the metric is,

$$g_{00} = -1 - 2\Psi(\vec{x}, t), \quad g_{0i} = 0, \quad g_{ij} = a^2\delta_{ij}(1 + 2\Phi(\vec{x}, t)), \tag{14}$$

where Ψ is Newtonian potential and Φ is the perturbation to spatial curvature, both of which are 1st order perturbations.

Now we simplify (13). Firstly, we note that P^0 is not an independent variable. In fact, for photon we have $0 = P^2 = g_{\mu\nu}P^\mu P^\nu = -(1+2\Psi)(P^0)^2 + p^2$, where $p^2 \equiv g_{ij}P^iP^j$. So we get,

$$P^0 = \frac{p}{\sqrt{1+2\Psi}} \simeq p(1-\Psi), \tag{15}$$

up to 1st order. Note that $\Psi < 0$ corresponds to overly dense region, so this equation says that a photon receives/loses energy, namely blue/red shifted, when travelling into/out of an overly dense region.

Now that P^0 is not independent, we should keep P^i in (13) only, and we will use the magnitude p and the direction $\hat{p}^i$ ($\delta_{ij}\hat{p}^i\hat{p}^j = 1$) as the variables, so that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial f}{\partial x^i} + \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\partial f}{\partial p} + \frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}\frac{\partial f}{\partial \hat{p}^i}.$$

At 0th order, the Bose-Einstein distribution of photon depends only on p, not on $\hat{p}^i$, so $\partial f/\partial\hat{p}^i$ is of 1st order. On the other hand, $\mathrm{d}\hat{p}^i/\mathrm{d}t$ is also of 1st order, since a photon does not change its direction in FRW background. Thus the last term on the right hand side in the above equation is of 2nd order and can be dropped in 1st order treatment.

$\mathrm{d}x^i/\mathrm{d}t$ term. By definition $P^\mu = \mathrm{d}x^\mu/\mathrm{d}t$ we have $\mathrm{d}x^i/\mathrm{d}t = P^i/P^0$. To represent this in terms of p and $\hat{p}^i$, we use $P^0 = p(1-\Psi)$, and assume $P^i = C\hat{p}^i$. The proportional constant can be found by $p^2 = g_{ij}P^iP^j = \delta_{ij}a^2(1+2\Phi)\delta_{ij}\hat{p}^i\hat{p}^jC^2 = a^2(1+2\Phi)C^2$, so $C = p(1-\Phi)/a, P^i = p\hat{p}^i(1-\Phi)/a$, and,

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\hat{p}^i}{a}(1+\Psi-\Phi). \tag{16}$$

Overly dense region has $\Psi < 0$ and $\Phi > 0$. Then the equation above states that photon decelerates in overly dense region. However, since $\partial f/\mathrm{d}x^i$ is of 1st order (BE distribution does not depend on coordinates), so we can further approximate the equation above to be $\mathrm{d}x^i/\mathrm{d}t \simeq \hat{p}^i/a$, so we have finally,

$$\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial f}{\partial x^i} = \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i}.$$

$\mathrm{d}p/\mathrm{d}t$ term. This term can be found by using the 0-component of geodesic equation,

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\mu\nu}P^\mu P^\nu. \tag{17}$$

The l.h.s reads,

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\Big[p(1-\Psi)\Big] = P^0\Big[\frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi) - p\frac{\mathrm{d}\Psi}{\mathrm{d}t}\Big],$$

so,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{P^0}(1+\Psi) = p\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{p}(1+2\Psi),$$

up to 1st order. The connection term can be worked out to be,

$$\Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{p} = (-1+2\Psi)\Big[-p\frac{\partial\Psi}{\partial t} - 2\frac{p\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} - p\Big(\frac{\partial\Phi}{\partial t} + H\Big)\Big].$$

So we get,

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = -H - \frac{\partial\Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}. \tag{18}$$

The three terms on the r.h.s. can be understood physically as follows. The first term shows the momentum gets lost due to expansion. The second term means that a gravitational potential well deepening with time ($\partial\Phi/\partial t > 0$) causes momentum decreasing. The third term means that a photon travelling into a potential ($\partial\Psi/\partial x^i < 0$) well acquires momentum.

Perturbation to the distribution function. To find the complete 1st order perturbed (13), we still have to expand the distribution function f. At 0th order, we have the usual BE distribution[1], $f^{(0)} = 1/(e^{p/T(t)} - 1)$, where the temperature T depends on time t due to cosmic expansion. So we write the 1st order perturbed form of f as,

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left(\frac{p}{T(t)\big(1+\Theta(\vec{x},\hat{p},t)\big)}\right) - 1\right]^{-1}. \tag{19}$$

Here we assume the temperature fluctuation $\Theta = \delta T/T$ does not depend on p. This is correct for Compton scattering because it does not change the magnitude of photon's momentum in effect. Now, expanding to 1st order, we have,

$$f = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta. \tag{20}$$

[1]The convention here, following Dodelson's, assigns the distribution to a single helicity state of photon, so there is no factor 2.

Zeroth-order Boltzmann's equation. Now we are ready to write down the zeroth order Boltzmann's equation for photons,

$$\frac{\mathrm{d}f}{\mathrm{d}t}\bigg|^{(0)} = \left(\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right) f^{(0)} = 0. \tag{21}$$

Here we don't include collision term on the r.h.s. because it is automatically made zero by 0th order distribution $f^{(0)}$. We are already able to read some physics from this equation. For example, the first term of this equation,

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\mathrm{d}T}{\mathrm{d}t}\frac{\partial f^{(0)}}{\partial T} = \frac{\mathrm{d}T}{\mathrm{d}t}\left(-\frac{p}{T}\frac{\partial f^{(0)}}{\mathrm{d}p}\right),$$

so this equation says,

$$\left(\frac{\mathrm{d}T/\mathrm{d}t}{T} + \frac{\mathrm{d}a/\mathrm{d}t}{a}\right)\frac{\partial f^{(0)}}{\partial p} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}T}{T} = -\frac{\mathrm{d}a}{a} \quad \Rightarrow \quad T \propto \frac{1}{a}.$$

Collisionless first-order Boltzmann's equation. At last we can write down the l.h.s. of 1st order perturbed part of photon's Boltzmann's equation, as,

$$\frac{\mathrm{d}f}{\mathrm{d}t}\bigg|^{(1)} = \left(\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right) f^{(1)} + \frac{\hat{p}^i}{a}\frac{\partial f^{(1)}}{\mathrm{d}x^i} - p\frac{\partial f^{(0)}}{\mathrm{d}p}\left(\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right),$$

where $f^{(1)} = -p(\partial f^{(0)}/\partial p)\Theta$. Keeping 1st order terms only, we get,

$$\frac{\mathrm{d}f}{\mathrm{d}t}\bigg|^{(1)} = -p\frac{\partial f^{(0)}}{\partial p}\left(\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right). \tag{22}$$

3.1.2 Photon: collision integral

Now we study the collision integral for photons' Boltzmann's equation, taking the collision to be Compton scattering with electron, $e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}\,') + \gamma(\vec{p}\,')$. We will ignore quantum statistical factors $(1 \pm f)$, because: 1) The stimulated emission factor $(1 + f)$ for photon is important only at 2nd order; 2) The Pauli blocking of electron $(1 - f_e)$ is never important after electron-positron annihilation, due to the tiny occupation number. Then, the collision integral becomes,

$$\begin{aligned} C[f(\vec{p})] = \frac{1}{2p}\int & \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\frac{\mathrm{d}^3 q'}{(2\pi)^3 2E_e(q')}\frac{\mathrm{d}^3 p'}{(2\pi)^3 2p'}\,|\mathcal{M}|^2 \\ & \times (2\pi)^4\delta^{(4)}(p + q - p' - q')\Big[f_e(\vec{q}\,')f(\vec{p}\,') - f_e(\vec{q})f(\vec{p})\Big]. \end{aligned} \tag{23}$$

The pre-factor $1/2p$ on the r.h.s. is from the fact that we are using $\mathrm{d}f/\mathrm{d}t$ rather than the covariant form $\mathrm{d}f/\mathrm{d}\lambda$, for the l.h.s. of Boltzmann's equation. This produces a factor $\mathrm{d}\lambda/\mathrm{d}t = (1+\Psi)/p \simeq 1/p$ for the r.h.s., where the last approximation is because the integral is already of 1st order.

We consider the case where electron is nearly nonrelativistic, so that $E_e(q) = m_e + q^2/2m_e$. now we try to carry out the collision integral.

The integral with d^3q' is straightforward with the aid of the $\delta^{(3)}$-function,

$$C[f(\vec{p})] = \frac{\pi}{8m_e^2 p}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{1}{p'}\delta\bigg(p' + \frac{(\vec{q}+\vec{p}-\vec{p}')^2}{2m_e} - p - \frac{q^2}{2m_e}\bigg)$$
$$\times |\mathcal{M}|^2\Big[f_e(\vec{q}+\vec{p}-\vec{p}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\Big].$$

To go further, we need some knowledge about typical kinetics of Compton scattering, given by $q \ll p, p'$ and $|\vec{p}' - \vec{p}| \sim p \sim T$. So we have,

$$E_e(q) - E_e(\vec{q}+\vec{p}-\vec{p}') \simeq \frac{(\vec{p}'-\vec{p})\cdot\vec{q}}{m_e} \sim \frac{Tq}{m_e} \sim Tv_b,$$

where the baryon's velocity $v_b \ll 1$. Here we speak of baryon velocity because electrons are tightly bounded with protons by Coulomb interaction. On the other hand, $E_q(q) \sim T \;\Rightarrow\; |E_e(q) - E_e(q')| \ll E_e(q) \;\Rightarrow$ We can expand $E_e(q')$ around $E_e(q)$,

$$\delta\bigg(p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q}+\vec{p}-\vec{p}')^2}{2m_e}\bigg) \simeq \delta(p-p') + \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}\frac{\partial}{\partial p'}\delta(p-p').$$

Then the collision integral becomes,

$$C[f(\vec{p})] = \frac{\pi}{8m_e^2 p}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\,f_e(\vec{q})\int\frac{\mathrm{d}^3p'}{(2\pi)^3p'}\,|\mathcal{M}|^2$$
$$\times\bigg[\delta(p-p') + \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}\frac{\partial}{\partial p'}\delta(p-p')\bigg]\Big(f(\vec{p}') - f(\vec{p})\Big).$$

It's time to call the amplitude $\mathcal{M}$ of Compton scattering. Let's ignore the effects of polarization for the moment, which do contribute to temperature anisotropy, and use the spin-averaged squared amplitude [2] ,

$$\Big\langle|\mathcal{M}|^2\Big\rangle_{\text{spin}} = 2e^4\bigg(\frac{p'}{p} + \frac{p}{p'} - \sin^2\langle\hat{p},\hat{p}'\rangle\bigg), \tag{24}$$

where $\langle\hat{p},\hat{p}'\rangle$ is the angle between $\hat{p}$ and $\hat{p}'$. In our case $p \simeq p'$, so $|\mathcal{M}|^2 \simeq 2e^4(1 + \cos^2\langle\hat{p},\hat{p}'\rangle) = 12m_e^2\sigma_T(1+\cos\langle\hat{p},\hat{p}'\rangle)$, where Thomson cross section $\sigma_T = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2}$ is the low energy limit of Compton scattering.

We can go on dealing with collision integral now,

$$C[f(\vec{p})] = \frac{\pi}{8m_e^2 p}\cdot 12\pi\sigma_T m_e^2\underbrace{\bigg(\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,f_e(\vec{q})\bigg)}_{n_e}$$

[2]See, e.g., Peskin & Schroeder, *An Introduction to Quantum Field Theory*, .

$$
\begin{aligned}
&\times\int\frac{\mathrm{d}^3p'\,(1+\cos^2\langle\hat{p},\hat{p}'\rangle)}{(2\pi)^3p'}\left[\delta(p-p')+\frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}\frac{\partial}{\partial p'}\delta(p-p')\right]\\
&\times\left[f^{(0)}(p')-f^{(0)}(p)-\left(p'\frac{\partial f^{(0)}}{\partial p'}\Theta(\hat{p}')-p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\right)\right]\\
=&\,\frac{3n_e\sigma_T}{16\pi p}\int_0^\infty\mathrm{d}p'\,p'\int\mathrm{d}\Omega'\,(1+\cos^2\langle\hat{p},\hat{p}'\rangle)\\
&\times\left[\delta(p-p')\Big(-p'\frac{\partial f^{(0)}}{\partial p'}\Theta(\hat{p}')+p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\Big)\right.\\
&\left.+(\vec{p}-\vec{p}')\cdot\underbrace{\vec{v}_b}_{\vec{q}/m_e}\left(\frac{\partial}{\partial p'}\delta(p-p')\right)\Big(f^{(0)}(p')-f^{(0)}(p)\Big)\right].
\end{aligned}
$$

The angular dependence here comes totally from $\Theta(\hat{p})$ and the factor $(1+\cos^2\langle\hat{p},\hat{p}'\rangle)$ in the integrand. In the context of cosmological perturbation theory, it is always assumed that $\Theta(\hat{p})$ has no azimuthal dependence, namely, the perturbation is axially symmetric[3] . We have seen that Compton amplitude is axially symmetric, so Θ will remain axially symmetric if it is originally so.

To carry the integral out, we define the ℓ-th multipole Θ_ℓ of $\Theta(\hat{p})$ by,

$$
\Theta_\ell(\vec{x},t)\equiv\frac{1}{4\pi(-\mathrm{i})^\ell}\int\mathrm{d}\Omega'\,\mathcal{P}_\ell(\hat{p}')\Theta(\hat{p}',\vec{x},t), \tag{25}
$$

where $\mathcal{P}_\ell(z)$ is the Legendre polynomial of ℓ-th order. We also need $1+\cos^2\langle\hat{p},\hat{p}'\rangle=\frac{4}{3}+\frac{2}{3}\mathcal{P}_2(\cos\langle\hat{p},\hat{p}'\rangle)$, as well as the formula,

$$
\mathcal{P}_\ell\big(\cos\langle\hat{p},\hat{p}'\rangle\big)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y_{\ell m}(\hat{p})Y^*_{\ell m}(\hat{p}').
$$

Due to axial symmetry, only $m=0$ term on the r.h.s. contributes. So, using $Y_{20}(\theta,\varphi)=-\sqrt{\frac{5}{4\pi}}\mathcal{P}_2(\cos\theta)$, we have,

$$
\begin{aligned}
C[f(\vec{p})]=&\,\frac{n_e\sigma_T}{p}\int_0^\infty\mathrm{d}p'\,p'\Big[\delta(p-p')\Big(-p'\frac{\partial f^{(0)}}{\partial p'}\Big(\Theta_0-\frac{1}{2}\mathcal{P}_2(\hat{p})\Theta_2\Big)+p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\Big)\\
&+\vec{p}\cdot\vec{v}_b\Big(\frac{\partial}{\partial p'}\delta(p-p')\Big)\Big(f^{(0)}(p')-f^{(0)}(p)\Big)\Big]\\
=&\,\frac{n_e\sigma_T}{p}\left[p\frac{\partial f^{(0)}}{\partial p}\Big(-\Theta_0+\frac{1}{2}\mathcal{P}_2(\hat{p})\Theta_2+\Theta(\hat{p})\Big)-(\vec{p}\cdot\vec{v}_b)p\frac{\partial f^{(0)}}{\partial p}\right]\\
=&-p\frac{\partial f^{(0)}}{\partial p}n_e\sigma_T\Big[\underbrace{\Theta_0}_{\text{monopole}}-\underbrace{\frac{1}{2}\mathcal{P}_2(\hat{p})\Theta_2}_{\text{quadrupole}}-\Theta(\vec{p})+\underbrace{\hat{p}\cdot\vec{v}_b}_{\text{dipole}}\Big].
\end{aligned} \tag{26}
$$

[3] See, e.g., C.-P. Ma and E. Bertschinger, *Astrophys. J.* **455** (1995) 7 [arXiv:astro-ph/9506072], at the end of Sec. 5.1.

This is our final result for photon's collision integral. It shows that Compton scattering leads to a rather simple distribution for photons, of which only monopole, dipole, and (perhaps) quadrupole are important.

3.1.3 Boltzmann's equation for photon

In previous two sub-subsections we have found both the l.h.s. and r.h.s. of Boltzmann's equation for photon in 1st order. Now we combine them, as,

$$\frac{\partial\Theta}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}=n_e\sigma_T\Big[\Theta_0-\frac{1}{2}\mathcal{P}_2(\hat{p})\Theta_2-\Theta(\vec{p})+\hat{p}\cdot\vec{v}_b\Big]. \tag{27}$$

To write it into a more convenient form, we use conformal time η instead. We will write $\dot{\Theta}\equiv \mathrm{d}\Theta/\mathrm{d}\eta$, etc. Then the above equation becomes,

$$\dot{\Theta}+\hat{p}^i\frac{\partial\Theta}{\partial x^i}+\dot{\Phi}+\hat{p}^i\frac{\partial\Psi}{\partial x^i}=n_e\sigma_T a\Big[\Theta_0-\frac{1}{2}\mathcal{P}_2(\hat{p})\Theta_2-\Theta(\vec{p})+\hat{p}\cdot\vec{v}_b\Big]. \tag{28}$$

We then go to $\vec{k}$-space, define,

$$\Theta(\vec{x})=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,e^{\mathrm{i}\vec{k}\cdot\vec{x}}\Theta(\vec{k}).$$

Small perturbations then means different modes do not mix. We further define the following two quantities that will be frequently used in subsequent sections.

1. Define $\mu\equiv\hat{k}\cdot\hat{p}$, namely the (cosine of the) angle between photon's direction ($\hat{p}$) and the direction of temperature's gradient ($\hat{k}$). We usually assume baryon's velocity is irrotational, namely $\vec{v}_b\parallel\vec{k}\ \Rightarrow\ \vec{v}_b\cdot\hat{p}=v_b\mu$.
2. Define the optical depth of photon, $\tau(\eta)\equiv\int_\eta^{\eta_0}\mathrm{d}\eta'\,n_e\sigma_T a$. Then $\dot{\tau}=-n_e\sigma_T a$. In late times n_e is small so $\tau\ll 1$, while in early times τ is large.

With these definitions, we write the following equation, as our final results for 1st order perturbed Boltzmann's equation of photon,

$$\boxed{\dot{\Theta}+\mathrm{i}k\mu\Theta+\dot{\Phi}+\mathrm{i}k\mu\Psi=-\dot{\tau}\Big(\Theta_0-\frac{1}{2}\mathcal{P}_2(\mu)\Theta_2-\Theta+\mu v_b\Big).} \tag{29}$$

3.1.4 Boltzmann's equation for cold dark matter

We take the W(eakly)I(nteracted)M(assive)P(article) assumption for dark matter. So these are nonrelativistic particle, which has $g_{\mu\nu}P^\mu P^\nu=-m^2$. Still using $(E,p,\hat{p}^i)$ as variables, we have then,

$$P^\mu=\Big(E(1-\Psi),\,p\hat{p}^i\,\frac{1-\Phi}{a}\Big),$$

and the Boltzmann's equation of DM's distribution function $f_{\rm dm}$ can be easily worked out to be,

$$\begin{aligned} 0 &= \frac{\mathrm{d}f_{\rm dm}}{\mathrm{d}t} = \frac{\partial f_{\rm dm}}{\partial t} + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial f_{\rm dm}}{\partial x^i} + \frac{\mathrm{d}E}{\mathrm{d}t}\frac{\partial f_{\rm dm}}{\mathrm{d}E} \\ &= \frac{\partial f_{\rm dm}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{\rm dm}}{\partial x^i} - \left[\frac{\mathrm{d}a/\mathrm{d}t}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial \Psi}{\partial x^i}\right]\frac{\partial f_{\rm dm}}{\mathrm{d}E}. \end{aligned} \tag{30}$$

The collision integral is simply zero because DM almost does not interact with anything. For the same reason, we do not have to assume a thermal distribution for it, but we do need to keep its velocity as the 1st order perturbation. So with the definition for number density $n_{\rm dm}$ and velocity $v^i_{\rm dm}$,

$$n_{\rm dm} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\rm dm}, \quad v^i_{\rm dm} = \frac{1}{n_{\rm dm}}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\rm dm}\frac{p\hat{p}^i}{E}, \tag{31}$$

let's firstly take the zeroth moment (monopole) of the above equation,

$$\begin{aligned} 0 = &\frac{\partial}{\partial t}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\rm dm} + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\rm dm}\frac{p\hat{p}^i}{E} \\ &- \left(\frac{\mathrm{d}a/\mathrm{d}t}{a} + \frac{\partial \Phi}{\partial t}\right)\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{p^2}{E}\frac{\partial f_{\rm dm}}{\partial E} - \frac{1}{a}\frac{\partial \Psi}{\partial x^i}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\hat{p}^i p. \end{aligned}$$

The first two terms are easy. The integral in the third term reads,

$$\begin{aligned} &\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{p^2}{E}\frac{\partial f_{\rm dm}}{\mathrm{d}E} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{p^2}{E}\frac{\mathrm{d}p}{\mathrm{d}E}\frac{\partial f_{\rm dm}}{\mathrm{d}p} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p\frac{\partial f_{\rm dm}}{\mathrm{d}p} \\ &= \frac{4\pi}{(2\pi)^3}\int_0^\infty \mathrm{d}p\, p^3\frac{\partial f_{\rm dm}}{\partial p} = -3\cdot\frac{4\pi}{(2\pi)^3}\int_0^\infty \mathrm{d}p\, p^2 f_{\rm dm} = -3n_{\rm dm}, \end{aligned}$$

while the fourth term is of second order and thus can be dropped. So we get the monopole equation,

$$\frac{\partial n_{\rm dm}}{\mathrm{d}t} + \frac{1}{a}\frac{\partial(n_{\rm dm}v^i_{\rm dm})}{\partial x^i} + 3\left(\frac{\mathrm{d}a/\mathrm{d}t}{a} + \frac{\partial \Phi}{\partial t}\right)n_{\rm dm} = 0, \tag{32}$$

which is actually the continuity equation for DM. Its zeroth order can be extracted as follows,

$$\frac{\partial n^{(0)}_{\rm dm}}{\mathrm{d}t} + 3\frac{\mathrm{d}a/\mathrm{d}t}{a}n^{(0)}_{\rm dm} = 0 \;\Rightarrow\; \frac{\mathrm{d}(n^{(0)}_{\rm dm}a^3)}{\mathrm{d}t} = 0 \;\Rightarrow\; n^{(0)}_{\rm dm} \propto \frac{1}{a^3},$$

which reproduces a familiar result. Now we define the 1st order perturbation to $n_{\rm dm}$ as $n_{\rm dm} = n^{(0)}_{\rm dm}\big(1 + \delta(\vec{x}, t)\big)$ with δ the density contrast, then the 1st order of the above continuity equation is,

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0. \tag{33}$$

where we have written $v^i_{\text{dm}} = v^i$ for simplicity. Now we take the first moment of (30), multiplying it by $\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p\hat{p}^j}{E}$,

$$\begin{aligned}0 =& \frac{\partial}{\partial t}\int\frac{\mathrm{d}^3p}{(2\pi)^3} f_{\text{dm}}\frac{p\hat{p}^j}{E} + \frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{\mathrm{d}^3p}{(2\pi)^3} f_{\text{dm}}\frac{p^2\hat{p}^i\hat{p}^j}{E^2}\\ &-\bigg(\frac{\mathrm{d}a/\mathrm{d}t}{a} + \underbrace{\frac{\partial\Phi}{\partial t}}_{\text{higher order}}\bigg)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p^3p^j}{E^2}\frac{\partial f_{\text{dm}}}{\partial E} - \frac{1}{a}\frac{\partial\Psi}{\partial x^i}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial f_{\text{dm}}}{\partial E}\frac{p^2\hat{p}^i\hat{p}^j}{E}.\end{aligned}$$

Again, the first term gives $\partial(n^{(0)}_{\text{dm}}v^j)/\partial t$, the second term contains two v^i's and thus can be dropped. The integral in the third term gives,

$$\begin{aligned}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p^3p^j}{E^2}\frac{\partial f_{\text{dm}}}{\partial E} &= \int\frac{\mathrm{d}\Omega\,\hat{p}^j}{(2\pi)^3}\int\mathrm{d}p\,\frac{p^5}{E^2}\frac{\mathrm{d}p}{\mathrm{d}E}\frac{\partial f_{\text{dm}}}{\partial p}\\ &= -\frac{\mathrm{d}\Omega\,\hat{p}^j}{(2\pi)^3}\int\mathrm{d}p\,f_{\text{dm}}\bigg(\frac{4p^3}{E} - \underbrace{\frac{p^5}{E^3}}_{\text{higher order}}\bigg) = -4n_{\text{dm}}v^j,\end{aligned}$$

and the integral in the fourth term reads,

$$\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial f_{\text{dm}}}{\partial E}\frac{p^2\hat{p}^i\hat{p}^j}{E} = \underbrace{\int\mathrm{d}\Omega\,\hat{p}^i\hat{p}^j}_{4\pi\delta^{ij}/3}\int\frac{\mathrm{d}p}{(2\pi)^3}p^3\frac{\partial f_{\text{dm}}}{\partial p} = -\delta^{ij}n_{\text{dm}}.$$

So the first moment of (30), up to 1st order, reads,

$$\frac{\partial(n^{(0)}_{\text{dm}}v^j)}{\partial t} + 4\frac{\mathrm{d}a/\mathrm{d}t}{a}n^{(0)}_{\text{dm}}v^j + \frac{n^{(0)}_{\text{dm}}}{a}\frac{\partial\Psi}{\partial x^j} = 0. \tag{34}$$

We further use the fact that $n^{(0)}_{\text{dm}} \propto a^{-3}$ to simplify the equation above, to get,

$$\frac{\partial v^j}{\partial t} + \frac{\mathrm{d}a/\mathrm{d}t}{a}v^j + \frac{1}{a}\frac{\partial\Psi}{\partial x^j} = 0. \tag{35}$$

Again, we use conformal time η and go to k-space. Using the irrotational assumption ($v^i = v\hat{k}^i$), we find the relevant 1st order equations (33) and (35), which form a close set of equations for 1st order perturbations (δ, v), becomes,

$$\boxed{\begin{cases}\dot{\delta} + \mathrm{i}kv + 3\dot{\Phi} = 0,\\ \dot{v} + \dfrac{\dot{a}}{a}v + \mathrm{i}k\Psi = 0.\end{cases}} \tag{36}$$

For DM the equations do not contain higher order moments because we keep velocity perturbation v^j only to linear term. For not-so-heavy DM, e.g. neutrino, we do need higher moments.

3.1.5 Boltzmann's equation for baryons

The unfortunate terminology of cosmology refers to both electrons and protons as baryons. In fact electrons and baryons are so tightly bounded by Coulomb attraction that we are able to take,

$$\frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} \equiv \delta_b, \quad \vec{v}_e = \vec{v}_p \equiv \vec{v}_b. \tag{37}$$

Then the Boltzmann's equations for electrons' distribution $f_e(\vec{x}, \vec{q}, t)$ and protons' distribution $f_p(\vec{x}, \vec{Q}, t)$ can be written schematically as,

$$\frac{\mathrm{d} f_e(\vec{x}, \vec{q}, t)}{\mathrm{d}t} = C_{\text{Coulomb}}[f_e] + C_{\text{Compton}}[f_e], \quad \frac{\mathrm{d} f_p(\vec{x}, \vec{Q}, t)}{\mathrm{d}t} = C_{\text{Coulomb}}[f_p]. \tag{38}$$

The Compton term for proton is absent because the cross section $\sigma \propto m^{-2}$ but $m_p \gg m_e$. Now we derive the equation for δ_b. The first step is to take the zeroth moment of above two equations. The l.h.s. of both equations are similar to the case of DM because they are all massive particles. So the equation for f_e gives,

$$\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial(n_e v_b^i)}{\partial x^i} + 3\left(\frac{\mathrm{d}a/\mathrm{d}t}{a} + \frac{\partial \Phi}{\partial t}\right) n_e = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \Big(C_{\text{Coulomb}}[f_e] + C_{\text{Compton}}[f_e]\Big).$$

The r.h.s. actually vanishes. To see this, we note that the the collision integral C_{Coulomb}, describing the process $e^-(q) + p(Q) \to e^-(q') + p(Q')$, is an integral over $(\vec{q}', \vec{Q}, \vec{Q}')$. Together with the additional integration over $\vec{q}'$, the integral measure is $\mathrm{d}^3 q \mathrm{d}^3 q' \mathrm{d}^3 Q \mathrm{d}^3 Q'$, and is clearly symmetric under exchange $(Q, q) \leftrightarrow (Q', q')$. On the other hand, the integrand of the collision integral contains a factor $f_e(q') f_p(Q') - f_e(q) f_p(Q)$, which is antisymmetric with the above exchange of momenta. Therefore $\int \mathrm{d}^3 q\, C_{\text{Coulomb}}[f_e(\vec{q})]$ is clearly zero. The same argument applies for the second term $\int \mathrm{d}^3 q\, C_{\text{Compton}}[f_e(\vec{q})]$, so the r.h.s. indeed vanishes. Then we have, in terms of conformal time η and in k-space, the following equation,

$$\boxed{\dot{\delta}_b + \mathrm{i} k v_b + 3\Phi = 0.} \tag{39}$$

From the experience with DM we know that the other equation can be found by taking the first moment of Boltzmann's equation above. Here, instead of multiplying the equation by $\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\vec{q}}{E}$, we choose $\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \vec{q}$. Then, comparing with DM's case, we should multiply the DM's results by $E(= m)$ to get baryons equation. We do this for the sum of two equations in (38). Now that $m_p \gg m_e$, the proton's equation dominates the l.h.s. in this case. Comparing with (34), we have,

$$\begin{aligned} & m_p \frac{\partial(n^{(0)} v_b^j)}{\partial t} + 4\frac{\mathrm{d}a/\mathrm{d}t}{a} m_p n_b^{(0)} v_b^j + \frac{m_p n_b^{(0)}}{a}\frac{\partial \Psi}{\partial x^j} \\ &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, q^j \Big(C_{\text{Coulomb}}[f_e] + C_{\text{Compton}}[f_e]\Big) + \int \frac{\mathrm{d}^3 Q}{(2\pi)^3}\, Q^j C_{\text{Coulomb}}[f_p]. \end{aligned}$$

Note that the sum $\int \mathrm{d}^3q\, q C_{\text{Coulomb}}[f_e] + \int \mathrm{d}^3Q\, Q C_{\text{Coulomb}}[f_p]$ is zero. This time the reason is that the two terms have a common integral measure $\mathrm{d}\vec{q}\mathrm{d}\vec{Q}\mathrm{d}\vec{q}'\mathrm{d}\vec{Q}'$, and the integral contains a δ-function of momentum conservation, $\delta^{(4)}(p+Q-p'-Q')$, so we can rewrite the factor $(q+Q)^j$ as $\frac{1}{2}(q+Q)+\frac{1}{2}(q'+Q')$. Then, again, this factor and the integral measure is symmetric under the exchange $(q,Q)\leftrightarrow(q',Q')$, while the rest part of the integrand is antisymmetric.

Using the same argument, we can rewrite the remaining Compton collision integral as,

$$\begin{aligned}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\, q^j C_{\text{Compton}}[f_e] &= \int \frac{\mathrm{d}^3q}{(2\pi)^3 2E_e(q)}\frac{\mathrm{d}^3q'}{(2\pi)^3 2E_e(q')}\frac{\mathrm{d}^3p}{(2\pi)^3 2p}\frac{\mathrm{d}^3p'}{(2\pi)^3 2p'}\, q^j|\mathcal{M}|^2\\ &\quad\times(2\pi)^4\delta^{(4)}(p+q-p'-q')\Big[f_e(\vec{q}')f(\vec{p}')-f_e(\vec{q})f(\vec{p})\Big]\\ &= -\int \frac{\mathrm{d}^3q}{(2\pi)^3 2E_e(q)}\frac{\mathrm{d}^3q'}{(2\pi)^3 2E_e(q')}\frac{\mathrm{d}^3p}{(2\pi)^3 2p}\frac{\mathrm{d}^3p'}{(2\pi)^3 2p'}\, p^j|\mathcal{M}|^2\\ &\quad\times(2\pi)^4\delta^{(4)}(p+q-p'-q')\Big[f_e(\vec{q}')f(\vec{p}')-f_e(\vec{q})f(\vec{p})\Big]\\ &= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\, p^j C[f(\vec{p})],\end{aligned}$$

where $C[f(\vec{p})]$ in the last line is the collision integral in photon's Boltzmann's equation that we have calculated in (26).

Now, dividing the equation by the baryon energy density $\rho_b = m_p n_b^{(0)}$, we get,

$$\frac{\partial(v_b^j)}{\partial t}+\frac{\mathrm{d}a/\mathrm{d}t}{a}v_b^j+\frac{1}{a}\frac{\partial\Psi}{\partial x^j} = -\frac{1}{\rho_b}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\, p^j C[f(\vec{p})].$$

Then we go to k-space, and multiply this equation by $\hat{k}^j$. Using $\hat{k}\cdot\vec{p} = p\mu$, the r.h.s. can be further rewritten as,

$$\begin{aligned}&-\frac{1}{\rho_b}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\, p\mu C_{\text{Compton}}[f_e(\mu)]\\ &=\frac{1}{\rho_b}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\, p\mu\cdot p\frac{\partial f^{(0)}}{\partial p}n_e\sigma_T\Big[\Theta_0-\frac{1}{2}\mathcal{P}_2(\mu)\Theta_2-\Theta(\mu)+v_b\mu\Big]\\ &=\frac{n_e\sigma_T}{\rho_b}\underbrace{\int_0^\infty\frac{\mathrm{d}p}{2\pi^2}p^4\frac{\partial f^{(0)}}{\partial p}}_{\text{int. by parts}\Rightarrow\,-4\rho_\gamma}\int_{-1}^{+1}\frac{\mathrm{d}\mu}{2}\,\mu\Big[\Theta_0-\frac{1}{2}\mathcal{P}_2(\mu)\Theta_2-\Theta(\mu)+v_b\mu\Big]\\ &=\frac{n_e\sigma_T}{\rho_b}(-4\rho_\gamma)\Big[\mathrm{i}\Theta_1+\frac{1}{3}v_b\Big].\end{aligned}$$

So the equation for v_b, when written with conformal time η, is,

$$\boxed{\dot{v}_b+\frac{\dot{a}}{a}v_b+\mathrm{i}k\Psi = \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}\Big(3\mathrm{i}\Theta_1+v_b\Big).} \tag{40}$$

3.1.6 Boltzmann's equation: a short summary

In this subsection we have derived equations governing the 1st order perturbations of photon, DM, and baryon. There are yet some missing pieces. Firstly we need equations for neutrino. This is straightforward if neutrinos are treated massless. In this case the corresponding equation can be easily got by replacing the variables in photon's equation and taking away all interaction terms. Let neutrino's "temperature fluctuation" be $\mathcal{N}$ (the corresponding variable for photon's Θ), we then have, following (29),

$$\boxed{\dot{\mathcal{N}} + \mathrm{i}k\mu\mathcal{N} = -\dot{\Phi} - \mathrm{i}k\mu\Psi.} \tag{41}$$

Another missing piece is photon's polarization. We will postpone this part to later sections. Here we simply introduce the variable Θ_P that describes the polarization, and write down, for completeness, the full set of Boltzmann's equation for 1st order perturbations of photons, baryons, DM, and massless neutrino.

$$\dot{\Theta} + \mathrm{i}k\mu\Theta = -\dot{\Phi} - \mathrm{i}k\mu\Psi - \dot{\tau}\Big[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi\Big], \tag{42a}$$

$$\dot{\Theta}_P + \mathrm{i}k\mu\Theta_P = -\dot{\tau}\Big[-\Theta_P + \frac{1}{2}\big(1 - \mathcal{P}_2(\mu)\big)\Pi\Big], \tag{42b}$$

$$\dot{\delta} + \mathrm{i}kv = -3\dot{\Phi}, \tag{42c}$$

$$\dot{v} + \frac{\dot{a}}{a}v = -\mathrm{i}k\Psi, \tag{42d}$$

$$\dot{\delta}_b + \mathrm{i}kv_b = -3\dot{\Phi}, \tag{42e}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -\mathrm{i}k\Psi + \frac{\dot{\tau}}{R}\big(v_b + 3\mathrm{i}\Theta_1\big), \tag{42f}$$

$$\dot{\mathcal{N}} + \mathrm{i}k\mu\mathcal{N} = -\dot{\Phi} - \mathrm{i}k\mu\Psi, \tag{42g}$$

with $\Pi \equiv \Theta_2 + \Theta_{P2} + \Theta_{P0}$, $R^{-1} \equiv 4\rho_\gamma^{(0)}/3\rho_b^{(0)}$.

3.2 Perturbed Einstein's equation

Now we derive the dynamical equations that govern the perturbations of spacetime. The l.h.s. of this equation is the Einstein tensor, and the r.h.s. is proportional to energy-momentum tensor.

3.2.1 Scalar perturbation

Still consider scalar perturbations at first. Then in conformal Newtonian gauge (14), we can derive the components of connection $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$ as,

$$\Gamma^0_{00} = \frac{\partial\Psi}{\partial t}, \qquad \Gamma^0_{0i} = \frac{\partial\Psi}{\partial x^i}, \qquad \Gamma^0_{ij} = a^2\delta_{ij}\Big[H(1-2\Psi+2\Phi) + \frac{\partial\Phi}{\partial t}\Big],$$

$$\Gamma^i_{00} = \frac{1}{a^2}\frac{\partial\Psi}{\partial x^i}, \quad \Gamma^i_{j0} = \delta_{ij}\Big(H + \frac{\partial\Phi}{\partial t}\Big), \quad \Gamma^i_{jk} = \delta_{ik}\frac{\partial\Phi}{\partial x^j} + \delta_{ij}\frac{\partial\Phi}{\partial x^k} - \delta_{jk}\frac{\partial\Phi}{\partial x^i}.$$

Then we can derive the components of Ricci tensor $R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}$, to be,

$$\begin{aligned}
R_{00} =& -3\Big(\frac{\mathrm{d}H}{\mathrm{d}t} + H^2\Big) + \frac{1}{a^2}\frac{\partial^2\Psi}{\partial x^i\partial x^i} - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\Big(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\Big),\\
R_{0i} =& -2\frac{\partial^2\Phi}{\partial t\partial x^i} + 2H\frac{\partial\Psi}{\partial x^i},\\
R_{ij} =& \, a^2\delta_{ij}\Big[\Big(\frac{\mathrm{d}H}{\mathrm{d}t} + 3H^2\Big)(1 - 2\Psi + 2\Phi) - H\frac{\partial\Psi}{\partial t} + \frac{\partial^2\Phi}{\partial t^2} + 6H\frac{\partial\Phi}{\partial t}\Big]\\
&- \delta_{ij}\frac{\partial^2\Phi}{\partial x^k\partial x^k} - \frac{\partial^2(\Psi+\Phi)}{\partial x^i\partial x^j}.
\end{aligned}$$

Then the Ricci scalar $R = g^{00}R_{00} + g^{ij}R_{ij}$ is,

$$\begin{aligned}
R = &\, 6\Big(\frac{\mathrm{d}H}{\mathrm{d}t} + 2H^2\Big) - 12\Big(\frac{\mathrm{d}H}{\mathrm{d}t} + 2H^2\Big)\Psi - 6H\frac{\partial\Psi}{\partial t}\\
&+ 6\frac{\partial^2\Phi}{\partial t^2} + 24H\frac{\partial\Phi}{\partial t} - \frac{2}{a^2}\frac{\partial^2(\Psi+2\Phi)}{\partial x^k\partial x^k}.
\end{aligned}$$

The 00-component and $0i$-component of Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ are,

$$\begin{aligned}
G^0{}_0 =& -3H^2 + 6H^2\Psi - 6H\frac{\partial\Phi}{\partial t} + \frac{2}{a^2}\frac{\partial^2\Phi}{\partial x^k\partial x^k},\\
G^0{}_i =& \, 2\frac{\partial^2\Phi}{\partial t\partial x^i} - 2H\frac{\partial\Psi}{\partial x^i}.
\end{aligned}$$

On the other hand, we do not need the complete form of $G^i{}_j$, but only its traceless and longitudinal part. Then the terms proportional to δ_{ij} will be irrelevant. So we will need,

$$G^i{}_j = (\cdots)\delta_{ij} - \frac{1}{a^2}\frac{\partial^2(\Phi+\Psi)}{\partial x^i\partial x^j}.$$

So much for the l.h.s. of Einstein's equation $G^\mu{}_\nu = 8\pi G T^\mu{}_\nu$. Now consider the energy-momentum tensor, which is given by,

$$T^\mu{}_\nu(\text{species } A) = g_A\int\frac{\mathrm{d}^3P}{(2\pi)^3}\sqrt{-g}\,\frac{P^\mu P_\nu}{P^0}f_A(\vec{x},\vec{p},t), \tag{43}$$

for a given species A described by the distribution f_A. Then, its 00, $0i$, and ij-components can be extracted, to 1st order in perturbation, to be,

$$\begin{aligned}
T^0{}_0 =& -g\int\frac{\mathrm{d}^3p}{(2\pi)^3}E(p)f(\vec{p},\vec{x},t),\\
T^0{}_j =& \, ga\int\frac{\mathrm{d}^3p}{(2\pi)^3}p\hat{p}_j(1+\Psi+\Phi)f(\vec{p},\vec{x},t),\\
T^j{}_k =& \, g\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p^2\hat{p}^j\hat{p}_k(1+2\Psi)}{E(p)}f(\vec{p},\vec{x},t).
\end{aligned}$$

Now, for nonrelativistic particles (DM and baryons), we have defined $\rho = \rho^{(0)}(1+\delta)$, so the corresponding contribution to $T^0{}_0$ at 1st order is simply $-\rho^{(0)}\delta$. For photon, $f = f^{(0)} - p(\partial f^{(0)}/\partial p)\Theta$, so,

$$T^0{}_0(\text{photon}) = -2\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p\Big(f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\Big) = -\rho_\gamma(1+4\Theta_0).$$

The factor 4 in the last expression can be easily understood from the relation $\rho \propto T^4$. For neutrino, we have the similar result $T^0{}_0(\text{neutrino}) = -\rho_\nu(1+4\mathcal{N}_0)$. So the 00 component of the 1st-order perturbed Einstein's equation reads,

$$6H^2\Psi - 6H\frac{\partial\Phi}{\partial t} + \frac{2}{a^2}\frac{\partial^2\Phi}{\partial x^k\partial x^k} = -8\pi G\Big(\rho_{\text{dm}}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\Big).$$

In conformal time and in k-space, this becomes,

$$\boxed{k^2\Phi + 3\frac{\dot a}{a}\Big(\dot\Phi - \frac{\dot a}{a}\Psi\Big) = 4\pi G a^2\Big(\rho_{\text{dm}}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\Big).} \tag{44}$$

Then we consider the traceless and longitudinal part of the ij-component of Einstein's equation, which can be picked out by projector $\partial_i\partial_j/\nabla^2 - \frac{1}{3}\delta_{ij}$. The l.h.s. reads,

$$\Big(\frac{\partial_i\partial_j}{\partial_k\partial_k} - \frac{1}{3}\delta_{ij}\Big)G^j{}_i = -\frac{2}{3a^2}\frac{\partial^2(\Phi+\Psi)}{\partial x^k\partial x^k}.$$

The r.h.s. contains $T^i{}_j$, which can be projected in k-space as,

$$\Big(\hat k_i\hat k_j - \frac{1}{3}\delta_{ij}\Big)T^j{}_i = \sum_i g_i\int\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{p^2\mu^2 - \frac{1}{3}p^2}{E_i(p)}(1+2\Psi)f_i(\vec p).$$

Now the integrand contains $(\mu^2 - \frac{1}{3}) = \frac{2}{3}\mathcal{P}_2(\mu)$, so only quadrupole contributes. But only the perturbations of photon and neutrino have quadrupole components, so, for photon,

$$\begin{aligned}\Big(\hat k_i\hat k_j - \frac{1}{3}\delta_{ij}\Big)T^j{}_i(\text{photon}) &= -2\int\frac{\mathrm{d}p}{2\pi^2}p^4\frac{\partial f^{(0)}}{\partial p}\int_{-1}^{1}\frac{\mathrm{d}\mu}{2}\frac{2\mathcal{P}_2(\mu)}{3}\Theta(\mu)\\ &= 2\cdot\frac{2\Theta_2}{3}\int\frac{\mathrm{d}p}{2\pi^2}p^4\frac{\partial f^{(0)}}{\partial p} = -\frac{8}{3}\rho^{(0)}\Theta_2.\end{aligned}$$

So the needed equation, written again with conformal time, is,

$$\boxed{k^2(\Phi+\Psi) = -32\pi G a^2\big(\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2\big).} \tag{45}$$

The above two equations, (44) and (45), do form a complete set of equations for two scalar perturbations of the metric. However, for completeness and also for future application, we derive the $0i$-component of Einstein's equation. In k-space, we multiply $\hat k^i$ with the $0i$-component of the energy-momentum tensor, which yields,

$$\hat k^i T^0{}_i = ga\int_0^\infty\frac{\mathrm{d}p}{2\pi^2}\,p\int_{-1}^{+1}\frac{\mathrm{d}\mu}{2}\mu(1+\Psi+\Phi)f(p,\mu,t).$$

As expected, only dipoles of distribution f survive the integration, which are of 1st order. Therefore the factor $(1+\Psi+\Phi)$ can be dropped, and we get,

$$\begin{aligned}\hat{k}^i T^0{}_i &= a\Big[\rho v + \rho_b v_b + 2\mathrm{i}\int_0^\infty \frac{\mathrm{d}p}{2\pi^2}\, p^4 \frac{\mathrm{d}}{\mathrm{d}p}\Big(f_\gamma^{(0)}\Theta_1 + f_\nu^{(0)}\mathcal{N}_1\Big)\Big]\\ &= a\Big(\rho v + \rho_b v_b - 4\mathrm{i}\rho_\gamma\Theta_1 - 4\mathrm{i}\rho_\nu\mathcal{N}_1\Big).\end{aligned}$$

Substitute this into the Einstein equation, we have,

$$2\mathrm{i}k\frac{\mathrm{d}\Phi}{\mathrm{d}t} - 2\mathrm{i}kH\Psi = 8\pi Ga\Big(\rho v + \rho_b v_b - 4\mathrm{i}\rho_\gamma\Theta_1 - 4\mathrm{i}\rho_\nu\mathcal{N}_1\Big)$$

Rewriting with conformal time η, we finally get,

$$\boxed{\dot{\Phi} - \frac{\dot{a}}{a}\Psi = \frac{4\pi Ga^2}{\mathrm{i}k}\Big(\rho v + \rho_b v_b - 4\mathrm{i}\rho_\gamma\Theta_1 - 4\mathrm{i}\rho_\nu\mathcal{N}_1\Big).} \tag{46}$$

3.2.2 Tensor perturbation

3.3 Initial condition

In previous subsections we developed a complete set of equations for perturbations. To solve them, we still need initial conditions. This involves physics at earlier time, much earlier than recombination or radiation-matter equality. We will see in this subsection that by studying physics at earlier time, we can relate initial values of different components by some equalities, so that all initial conditions are finally reduced to the initial condition for spatial curvature perturbation Φ. However, the initial condition for Φ cannot be determined in traditional big-bang cosmology. One can only accept it, or appeal to some conjectured theory at much higher energies such as inflation theory.

Recall that we have 9 variables for scalar perturbations, including Θ and Θ_P for photon, $\mathcal{N}$ for neutrino, δ and v for (DM-dominated) matter, δ_b and v_b for baryonic matter, and finally, Φ and Ψ for metric.

At early times, most modes we are interested with are outside the horizon, namely $k\eta \ll 1$. In equations for matter perturbations (42), terms like $\dot{\Theta}$ are of order Θ/η, and is much larger than terms like $k\Theta$. So all similar terms multiplied by k can be neglected. For the same reason, the photon and neutrino perturbations are nearly smooth and isotropic within a causally connected region, so all of their higher moments can be neglected, leaving zeroth monopoles only. These considerations imply,

$$\dot{\Theta}_0 + \dot{\Phi} = 0, \quad \dot{\mathcal{N}}_0 + \dot{\Phi} = 0, \quad \dot{\delta} + 3\dot{\Phi} = 0, \quad \dot{\delta}_b + 3\dot{\Phi} = 0. \tag{47}$$

Then we turn to Einstein's equation. At early times k^2 term and matter energy densities can be dropped from (44), which then becomes,

$$\frac{\dot{a}}{a}\Big(\dot{\Phi} - \frac{\dot{a}}{a}\Psi\Big) = \frac{16\pi G}{3}a^2\Big(\rho_\gamma\Theta_0 + \rho_\nu\mathcal{N}_0\Big).$$

At early times radiation dominates, and thus $a \sim \sqrt{t} \;\Rightarrow\; a \sim \eta \;\Rightarrow\; \dot{a}/a = 1/\eta$, so we have,

$$\frac{1}{\eta}\left(\dot{\Phi} - \frac{\Psi}{\eta}\right) = \frac{16\pi G}{3}a^2\Big(\rho_\gamma \Theta_0 + \rho_\nu \mathcal{N}_0\Big) = \frac{16\pi G\rho}{3}a^2\left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}\mathcal{N}_0\right).$$

Meanwhile we have $H = a^{-1}\mathrm{d}a/\mathrm{d}t = 1/(a\eta)$, and thus the Friedmann's equation $(a\eta)^{-2} = 8\pi G\rho/3$. Substituting this into the r.h.s. of above equation, we have,

$$\eta\dot{\Phi} - \Psi = 2\Big[(1-f_\nu)\Theta_0 + f_\nu \mathcal{N}_0\Big],$$

where we have defined $f_\nu \equiv \rho_\nu/(\rho_\gamma + \rho_\nu)$ to be the fraction of neutrino energy density from total radiation. Now taking derivative of this equation with conformal time, we get,

$$\dot{\Phi} + \eta\ddot{\Phi} - \dot{\Psi} = -2\dot{\Phi}.$$

Now we use (45), which states that $\Psi = -\Phi$ if quadrupole can be neglected. This is true for photon because photon's quadrupole is highly suppressed due to Compton scattering. However, neutrino's quadrupole may not be negligibly small and may give some corrections. Setting this issue aside, we simply take $\Psi = -\Phi$ in the following, so that the above equation becomes $\eta\ddot{\Phi} + 4\dot{\Phi} = 0$. We immediately get two modes, $\Phi =$ const. and $\Phi = \eta^{-3}$. The latter is a decaying mode, so is not important. The former, on the other hand, is a constant, and can carry the initial conditions for perturbations of our universe. Then we have,

$$\Phi = 2\Big[(1-f_\nu)\Theta_0 + f_\nu \mathcal{N}_0\Big],$$

which means Θ_0 and $\mathcal{N}_0$ are also constants. Usual mechanisms generating these modes do not distinguish between photon and neutrino, so we will set $\Theta_0 = \mathcal{N}_0$ at early times, and thus,

$$\Theta_0(k,\eta_{\rm ini}) = \mathcal{N}_0(k,\eta_{\rm ini}) = \frac{1}{2}\Phi(k,\eta_{\rm ini}). \tag{48}$$

For massive particles, using (47), we have $\delta_i = 3\Theta_0 + \text{const.}$ where i can be either DM or baryon. For so-called adiabatic perturbations, these constants are zero. The adiabatic perturbation is such that the ratio of number densities $n_{\rm dm}/n_\gamma$ is constant in both space and time. Note that

$$\frac{n_{\rm dm}}{n_\gamma} = \frac{n_{\rm dm}^{(0)}}{n_\gamma^{(0)}}\frac{1+\delta}{1+3\Theta_0} = \frac{n_{\rm dm}^{(0)}}{n_\gamma^{(0)}}\Big(1+\delta-3\Theta_0\Big),$$

where the prefactor of zeroth order is indeed a constant in space and time. Thus the whole ratio is a constant only when $\delta = 3\Theta_0$. This is the initial condition we will use in the following. There are also non-adiabatic initial conditions, namely the so-called isocurvature perturbations, which we won't consider here. Thus, for DM and baryons, we have,

$$\delta(k,\eta_{\rm ini}) = 3\Theta_0(k,\eta_{\rm ini}), \quad \delta_b(k,\eta_{\rm ini}) = 3\Theta_0(k,\eta_{\rm ini}). \tag{49}$$

4 Inflation Theory

5 Evolution of Matter Distribution

In previous sections we have derived the dynamical equations governing the evolution of perturbations. From now on we are going to solve them. In this section we will concentrate on matter distribution, dominated by DM. In next section we will study CMB anisotropy.

The basic picture is as follows. The matter perturbations begin to evolve with time after entering the horizon. The larger the scale of perturbation, the later it enters the horizon. It is expectable that entering the horizon before or after the epoch of radiation-matter equality a_{eq} makes a big difference. Basically, for large modes entering the horizon after a_{eq}, the gravitational potential $\varPhi$ will evolve smoothly to a constant value; for small modes entering horizon before a_{eq}, they feel the effect of hot photons right after entering the horizon, and thus begin to oscillate, which are gradually damped, and finally reach a constant.

Although gravitational potential $\varPhi$ reaches constant on both large and small scales, the matter density contrast δ begin to increase due to the attractive potential $\varPhi$. Then it reaches nonlinear regime, and finally forms galaxies, as the large scale structure we see today.

6 Cosmological Microwave Background

References