Nonlinear Realization of Global Symmetries

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Abstract

1 Introduction

The concept of symmetry is of central importance in modern physics. Plenty information can be extracted from the symmetry of a physical system. In field theory, the symmetry plays an essential role in defining some very basic concept like particle, charge, etc. Symmetry also put strong constraint on the properties of fields/particles, as well as their interactions and correlations.

In quantum field theory, Wigner's theorem tells us that symmetry transformations act as unitary or anti-unitary operators acting on the Hilbert space. Under the action of symmetry operators, states in Hilbert space are arranged into representations (or more rigorously, projective representations) of the symmetry group G. There are basically two ways of symmetry realization in Hilbert space. One is such that the vacuum state is invariant under the action of all symmetry transformations, which is known to be the Wigner-Weyl realization, the other contains degenerate vacua, which can transforms into each other under some of the symmetry transformation. This is known as Nambu-Goldstone realization.

It can be shown that in the Wigner-Weyl realization, excited states above vacuum in Hilbert space are organized into linear representations of the symmetry group. Furthermore, the field operators corresponding to these states lie in the same linear representations. Thus, from the viewpoint of fields, we may say the symmetry is linearly realized in Wigner-Weyl realization. On the other hand, in Nambu-Goldstone realization, the vacua are invariant only under the action of a subgroup H of full symmetry group G. When acting a group element outside H on a vacuum state, it will move to another vacuum. Intuitively, if this action is applied locally, i.e., with different group elements on different spacetime points, a special massless excitation will be generated, which is called the Goldstone mode. The existence of Goldstone modes Nambu-Goldstone realization is established by the famous Nambu-Goldstone's theorem. Besides Goldstone modes, there can of course be other particle states in the theory. However, the states within the same representation of the full symmetry group G may have different mass. Thus we see that the spectrum of Nambu-Goldstone realization is very different from Wigner-Weyl realization. Meanwhile, the field operators corresponding to these states (both Goldstone modes and other particles) do not form linear representation of G. Therefore we may say the symmetry is nonlinearly realized in Nambu-Goldstone realization.

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In the language of Lagrangian formulation of field theory, symmetry is reflected in the invariance of action S under the action of group G, no matter it is realized linearly or nonlinearly. For linear realization, the construction of such action is relatively simple: one can easily build a G-invariant Lagrangian $\mathscr{L}$ with covariant field operators[1]. For nonlinear realization, the construction becomes far more nontrivial. In this note, we will study the construction of low energy effective theories with nonlinearly realized symmetry.

The standard method for constructing G-invariant Lagrangian with nonlinear realization is known as coset construction. The name is because, as we will see in the following, the Goldstone modes can be identified as points in the coset space G/H where H is the linearly realized subgroup of full symmetry G.

2 Coset Construction for Global Internal Symmetries

As a first step, we consider nonlinearly realized global internal symmetries in this section. From this statement it is clear that we will assume manifest Poincaré symmetry of spacetime.

2.1 General formulation

Consider the theory with symmetry described by a semi-simple and simply connected Lie group G, and with a sub-Lie group H of G being linearly realized. The claim of coset construction is that the Lagrangian of the theory is constructed from Goldstone fields and other fields linearly representing H. The Goldstone fields in this case are bosonic functions of spacetime points, and taking values in the coset space G/H. In the following, only local properties of Lie groups are relevant, thus we can consider the corresponding Lie algebra instead. Let the Lie algebra corresponding G and H be $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We denote a set of independent generators in $\mathfrak{h}$ by V^a with $a = 1, \cdots, \dim\mathfrak{h}$. Then, a basis of $\mathfrak{g}$ can be formed with V^a and remaining independent generators A^i where $\alpha = 1, \cdots \dim\mathfrak{g} - \dim\mathfrak{h}$. The basis can always be chosen to be orthonormal and such that the structure constants are totally antisymmetric, namely $\operatorname{tr}(V^aV^b) = \delta^{ab}$, $\operatorname{tr}(A^iA^j) = \delta^{ij}$, and $\operatorname{tr}(V^aA^i) = 0$. Furthermore, $[V^a, V^b] = \mathrm{i}f^{abc}V^c$, $[A^i, A^j] = \mathrm{i}f^{ijk}A^k + \mathrm{i}f^{ija}V^a$ and $[V^a, A^i] = \mathrm{i}f^{aij}A^j$.

The Goldstone fields $\pi^i = \pi^i(x)$, as claimed, parameterize the coset space. Therefore, we may write a representative element $U(\pi)$ in G/H to be

$$U(\pi) = e^{\mathrm{i}\pi\cdot A}. \tag{1}$$

Under the left action of an element $g \in G$, $U(\pi)$ will in general not be in of the form $e^{\mathrm{i}\tilde{\pi}\cdot A}$, but develop V-dependent terms in the exponent. Therefore, to bring $gU(\pi)$ back to the standard parameterization above, we need a compensating transformation $h = h(g, \pi)$ acting from right, which depends on both the group element g and the Goldstone field π^i. Note that $\pi^i = \pi^i(x)$ is a function of spacetime, so is $h(g, \pi)$. As a result, we have

$$g: \ U(\pi) \to U(\tilde{\pi}) = gU(\pi)h^\dagger(g, \pi). \tag{2}$$

[1]In fact, the G-invariance of action S does not imply the G-invariance of $\mathscr{L}$, but only a G-invariant $\mathscr{L}$ up to a total divergence. This is a very interesting issue in itself. However, we will ignore this possibility for most of the following study

This expression determines the transformation behavior of Goldstone fields π^i. It is in general a complicated nonlinear transformation law. As group realization, we need this transformation to be associative, and the identity element in g is trivially represented. That is, we demand $h(g_2g_1,\pi) = h(g_2,\tilde{\pi}(g_1,\pi))h(g_1,\pi)$ and $h(1,\pi) = 1$. In particular, the linear realization of H requires the compensating transformation $h(g,\pi)$ satisfying $h(h,\pi) = h$ for all $h \in H$, so that $h : U(\pi) = hU(\pi)h^\dagger$, which implies

$$h : \pi^i A^i \to \pi^i hA^i h^\dagger = \pi^i D^{ij}(h)A^j. \tag{3}$$

That is, $h(h,\pi)$ has no π-dependence, and π^i field transforms linearly under the action of H.

The analysis above shows that the action of symmetry group G on $U(\pi)$ is realized by a global and linear left action, together with a nonlinear local transformation through $h(g,\pi) \in H$. In particular, the right action of H becomes global and linear when $g \in G$. It can be imagined that the transformation behavior of π^i fields is quite complicated, but it is hopeful to use $U(x)$ to construct G-invariant quantities. Clearly, such quantities must contain differential forms, since otherwise the only invariant will be $U^\dagger U$, which is a constant. Therefore, we should consider quantities such like $\mathrm{d}U$. However, one immediately see that $\mathrm{d}U$ is not covariant, and transforms under G-action as $\mathrm{d}U \to g\mathrm{d}Uh^\dagger + gU\mathrm{d}h^\dagger$ since the right action h is local. The solution to this problem is familiar, we simply introduce a gauge connection 1-form $\mathcal{A}$, so that the 1-form $\mathrm{d}U - \mathrm{i}U\mathcal{A} \equiv \mathrm{D}U$ transforms covariantly, namely, $\mathrm{D}U \to g\mathrm{D}Uh^\dagger$. Furthermore, since $h(g,\pi)$ depends on spacetime coordinates through π^i field, it is reasonable to imagine that the gauge connection 1-form $\mathcal{A}$ can also be constructed from π^i. An elegant way of this construction is to consider the 1-form $\omega = -\mathrm{i}U^\dagger(\pi)\mathrm{d}U(\pi)$, known as Maurer-Cartan form in mathematics. It is straightforward to see that ω transforms as

$$\omega \to h\omega h^\dagger - \mathrm{i}h\mathrm{d}h^\dagger, \tag{4}$$

just in the way we want. Being a 1-form in the dual space $\mathfrak{g}^*$ of the original Lie algebra $\mathfrak{g}$, the Maurer-Cartan form can be decomposed in the dual basis $\{V_a^*, A_i^*\}$ of $\mathfrak{g}^*$, or simply the generator basis $\{V^a, A^i\}$ of $\mathfrak{g}$ through the identification of Killing form, as

$$\omega = \omega_V + \omega_A = \omega_V^a V^a + \omega_A^i A^i. \tag{5}$$

Then the longitudinal (V) and transverse (A) part of Maurer-Cartan form transform as a gauge connection and a linear representation, respectively. That is, $\omega_V \to h\omega_V h^\dagger - \mathrm{i}h\mathrm{d}h^\dagger$ and $\omega_A \to h\omega_A h^\dagger$. Hence ω_V is simply the needed gauge connection 1-form $\mathcal{A}$. More explicitly, suppose the theory also contains a field variable ψ, which transforms linearly under H as $\psi \to D(h)\psi$. Then the G-action on ψ can be realized nonlinearly through $h = h(g,\pi)$ to be $\psi \to D(h(g,\pi))\psi$. As a consequence, the original global symmetry transformation h is promoted to a local one, and the corresponding gauge connection is just given by ω_V. One can easily check that $\mathrm{D}\psi = (\mathrm{d}+\mathrm{i}\omega_V^a D(T^a))\psi$ transforms covariantly. In particular, the object ω_A contains exactly the covariant differential of $U(\pi)$, that is, $\omega_A = -\mathrm{i}U^\dagger(\pi)\mathrm{D}U(\pi)$. Indeed,

$$-\mathrm{i}U^\dagger\mathrm{D}U = -\mathrm{i}U^\dagger(\mathrm{d}U - \mathrm{i}U\omega_V) = \omega - \omega_V = \omega_A. \tag{6}$$

2.2 Uniqueness of nonlinear realizations

2.3 Example 1: $SU(2) \to U(1)$

As the first illustration of coset construction, we consider a toy model with global internal symmetry $SU(2) \to U(1)$. The $SU(2)$ group is generated by three generators T_i ($i = 1, 2, 3$) satisfying the commutation relation:

$$[T_i, T_j] = \mathrm{i}\epsilon_{ijk}T_k. \tag{7}$$

Let T_3 be the generator of the linearly realized $U(1)$ subgroup. Then the two dimensional coset space $SU(2)/U(1)$ can be parameterized by variables π^a ($a = 1, 2$) in the standard way, $e^{\mathrm{i}\pi^a T_a} = e^{\mathrm{i}(\pi^1 T_1 + \pi^2 T_2)}$, while the group $SU(2)$ can be parameterized as $e^{\mathrm{i}\pi^a T_a} e^{\mathrm{i}\sigma T_3}$. We will allow $\pi^a = \pi^a(x)$ to be functions of spacetime coordinates, more rigorously, we are actually considering the coset bundle over the spacetime manifold. The fields $\pi^a(x)$ will play the role of Goldstone field in the effective Lagrangian formulation. To find out how they transform under the group action $g \in SU(2)$, we note that the coset variable $U(\pi) = e^{\mathrm{i}\pi^a T_a}$ transforms under the left action of g as

$$U(\pi) \to gU(\pi)h(\pi, g), \tag{8}$$

where h belongs to the unbroken $U(1)$ and is necessary because $gU(\pi)$ is in general not of the form $e^{\mathrm{i}\pi^a T_a}$.

A crucial observation of coset construction is that the Maurer-Cartan form $\omega = -\mathrm{i}U^{-1}(\pi)\mathrm{d}U(\pi)$ transforms under the left action of $g \in SU(2)$ according to

$$\omega \to h^{-1}(\pi, g)\omega h(\pi, g) - \mathrm{i}h^{-1}(\pi, g)\mathrm{d}h(\pi, g). \tag{9}$$

This Lie-algebra valued 1-form can always be expanded in terms of generators,

$$\omega = \omega_A^a T_a + \omega_V T_3. \tag{10}$$

Therefore we see that the transverse part ω_A transforms covariantly under a local transformation $h \in U(1)$ (local in the sense that it depends on spacetime coordinates implicitly through its dependence on π^a), while the longitudinal part ω_V transforms as a connection. Therefore, ω_A can be used as a building block of the effective Lagrangian.

On the other hand, there could also be some fields, say ψ, in the Lagrangian that realize the unbroken $U(1)$ linearly, namely, ψ transforms under a linear representation of $h \in U(1)$, $\psi \to D(h)\psi$. In order that this piece also realize the whole $SU(2)$ symmetry (though nonlinearly), we promote this transformation to a local one, $\psi \to D\big(h(\pi, g)\big)\psi$. Then the needed connection due to the local transformation is provided by ω_V.

For $SU(2)/U(1)$, we parameterize an infinitesimal group element to be $g = e^{\mathrm{i}\xi^a T_a} e^{\mathrm{i}\eta T_3}$, and the compensating local transformation can be written as $h = e^{\mathrm{i}\sigma T_3}$. Then expanding all these quantities up to linear order in group generators, we have

$$1 + \mathrm{i}(\pi^a + \delta\pi^a)T_a = (1 + \mathrm{i}\xi^b T_b)(1 + \mathrm{i}\eta T_3)(1 + \mathrm{i}\pi^c T_c)(1 + \mathrm{i}\sigma T_3). \tag{11}$$

Then up to the first nontrivial order, we have

$$\delta\pi^1 = \xi^1 + 2\eta\pi^2, \qquad \delta\pi^2 = \xi^2 - 2\eta\pi^1, \qquad \sigma = -\eta + \xi^1\pi^2 - \xi^2\pi^1. \tag{12}$$

Now we calculate the Maurer-Cartan form explicitly. With $U(\pi) = e^{\mathrm{i}(\pi^1 T_1 + \pi^2 T_2)}$, we have

$$\mathrm{d}U(\pi) = e^{\mathrm{i}(\pi^1 T_1 + \pi^2 T_2)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \,\mathrm{ad}\,^n_{\mathrm{i}\pi^a T_a}(\mathrm{i}T_b)\mathrm{d}\pi^b. \tag{13}$$

It is straightforward to show that

$$\mathrm{ad}\,^{2n}_{\mathrm{i}\pi^a T_a}(\mathrm{i}T_1) = \mathrm{i}\big(-|\pi|^2\big)^{n-1}\Big(-(\pi^2)^2 T_1 + \pi^1\pi^2 T_2\Big), \tag{14}$$

$$\mathrm{ad}\,^{2n-1}_{\mathrm{i}\pi^a T_a}(\mathrm{i}T_1) = \mathrm{i}\big(-|\pi|^2\big)^{n-1}\pi^2 T_3, \tag{15}$$

$$\mathrm{ad}\,^{2n}_{\mathrm{i}\pi^a T_a}(\mathrm{i}T_2) = \mathrm{i}\big(-|\pi|^2\big)^{n-1}\Big(\pi^1\pi^2 T_1 - (\pi^1)^2 T_2\Big), \tag{16}$$

$$\mathrm{ad}\,^{2n-1}_{\mathrm{i}\pi^a T_a}(\mathrm{i}T_2) = -\,\mathrm{i}\big(-|\pi|^2\big)^{n-1}\pi^1 T_3, \tag{17}$$

in which $n \geq 1$ and $|\pi| \equiv \sqrt{(\pi^1)^1 + (\pi^2)^2}$. Then the series can be summed into a closed form, and we get finally

$$\begin{aligned}\omega = &\left[\left(1 - \frac{|\pi| - \sin|\pi|}{|\pi|^3}(\pi^2)^2\right)\mathrm{d}\pi^1 + \left(\frac{|\pi| - \sin|\pi|}{|\pi|^3}\pi^1\pi^2\right)\mathrm{d}\pi^2\right]T_1 \\ &+ \left[\left(\frac{|\pi| - \sin|\pi|}{|\pi|^3}\pi^1\pi^2\right)\mathrm{d}\pi^1 + \left(1 - \frac{|\pi| - \sin|\pi|}{|\pi|^3}(\pi^1)^2\right)\mathrm{d}\pi^2\right]T_2 \\ &+ \frac{1 - \cos|\pi|}{|\pi|^2}\big(\pi^2\mathrm{d}\pi^1 - \pi^1\mathrm{d}\pi^2\big)T_3.\end{aligned} \tag{18}$$

Now that we have found the explicit form of ω_A^a and ω_V, we can use them to construct effective Lagrangian. Firstly, the kinetic for the Goldstone fields π^a is given by the term $\omega_A^a \wedge \omega_A^a \wedge e^b \wedge e^b$, where e^b is the spacetime vierbein 1-form. This term gives

$$\begin{aligned}&\frac{1}{2}\left(1 - \frac{|\pi| - \sin|\pi|}{|\pi|^3}(\pi^2)^2\right)^2(\partial_\mu\pi^1)^2 + \frac{1}{2}\left(1 - \frac{|\pi| - \sin|\pi|}{|\pi|^3}(\pi^1)^2\right)^2(\partial_\mu\pi^2)^2 \\ &+ \frac{1}{2}\left(\frac{|\pi| - \sin|\pi|}{|\pi|^3}\pi^1\pi^2\right)^2\Big((\partial_\mu\pi^1)^2 + (\partial_\mu\pi^2)^2\Big) + \frac{|\pi|^2 - \sin^2|\pi|}{|\pi|^4}\pi^1\pi^2\partial_\mu\pi^1\partial^\mu\pi^2.\end{aligned} \tag{19}$$

Now let's have a close look at this term for small π^a. At the leading order, we have the correct kinetic term

$$\frac{1}{2}(\partial_\mu\pi^a)^2. \tag{20}$$

Then, at the order of π^4, we have,

$$-\frac{1}{6}\big(\pi^2\partial_\mu\pi^1 - \pi^1\partial_\mu\pi^2\big)^2. \tag{21}$$

For a Dirac spinor ψ transforms under the fundamental representation of $SU(2)$, we can write down the kinetic term to be

2.4 Example 2: strong interaction at low energies

3 Nonlinearly Realized Spacetime Symmetries

3.1 General formulation

The basic strategy for constructing theory with nonlinear spacetime symmetry is similar to the case of internal symmetry. But there are two new features. One is that the spacetime coordinates

x^μ transform under spacetime translation P^μ nonlinearly. Indeed, the coordinate shift $x^\mu \to x^\mu + a^\mu$ looks like Goldstone bosons under broken symmetry transformation. Thus we should in any case treat the translation P^μ as broken generators. Then the spacetime itself can be viewed as the coset space $ISO(3,1)/SO(3,1)$, namely, Poincaré/Lorentz[2]. Another new feature is that the number of independent Goldstone fields π^i can be smaller than the number of broken generators, a phenomenon we all familiar with. For instance, when conformal symmetry spontaneously breaks to Poincaré symmetry, the system usually has only 1 Goldstone mode, namely the dilaton, while there are 5 broken generators — 1 dilatation plus 4 special conformal transformations. Roughly speaking, the mismatch is because the formulation of Goldstone field in coset space needs localized symmetry, and spacetime symmetries would mix with each other when localized.

Let the full (spacetime+internal) symmetry group be G, and we assume that $ISO(3,1)$ is an unbroken subgroup of G. We denote broken generators of G by A^i, translations by P^μ, Lorentz transformations by $J^{\mu\nu}$, and the rest of unbroken generators by V^a. Once again, the coset space can be parameterized by

$$U(\pi, x) = e^{\mathrm{i}x_\mu P^\mu} e^{\mathrm{i}\pi^i(x)A^i}. \tag{22}$$

The left G-action on $U(\pi, x)$ then reads

$$g: \; U(\pi(x), x) = gU(\pi'(x'), x')h^{-1}(g, \pi(x)), \tag{23}$$

where $h = h(g, \pi(x))$ belongs to the unbroken group[3], and can be parameterized by $e^{\mathrm{i}u^a V^a} e^{\mathrm{i}\omega_{\mu\nu} J^{\mu\nu}/2}$. As for the internal symmetry, we still require that $h = h(g, \pi(x))$ to keep the group product rule, and that $h(h, \pi(x)) = h$. In particular, for $g \in ISO(3,1)$ with Poincaré transformation parameter (Λ, a) such that $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, we have $g : U(\pi, x) = gU(\pi, x)g^{-1} = U(\pi'(x'), x')$ with $\pi'^i(x') = D^{ij}(g)\pi^j(x)$ and $D^{ij}(g)$ belongs to a linear representation of Poincar'e group. Thus the Goldstone field transforms linearly under the action of Poincaré group, a direct consequence of the fact that Poincar'e group is unbroken.

To construct invariants, we turn to Maurer-Cartan form again, which is now given by

$$\begin{aligned} \omega &= -\,\mathrm{i}U(\pi, x)\mathrm{d}U(\pi, x) = \omega_P + \omega_J + \omega_V + \omega_A \\ &= \omega_P^\mu P_\mu + \tfrac{1}{2}\omega_J^{\mu\nu} J_{\mu\nu} + \omega_V^a V^a + \omega_A^i A^i. \end{aligned} \tag{24}$$

From the transformation law of $U(\pi, x)$ we can deduce the behavior of Maurer-Cartan form under the G-action. In terms of components, it can be represented as

$$\omega_P \to h\omega_P h^{-1}, \qquad \omega_A \to h\omega_A h^{-1}, \qquad \omega_J + \omega_V \to h(\omega_J + \omega_V)h^{-1} + h\mathrm{d}h^{-1}. \tag{25}$$

Again, the components corresponding to “broken” generators transform covariantly, while the components corresponding to “unbroken” transforms like gauge connections. The expression above shows that ω_P and ω_A transform independently, which is based on assumed commutation relations such as $[J, A] \sim A$, $[V, A] \sim A$ and $[P, Z] \sim P$. The first two commutators represent simply the fact that broken generators form linear representations of unbroken symmetries. The third one may not be true in general.

[2]When we mention symmetry group like Poincaré group or Lorentz group, we always mean its connected component containing identity.

[3]Note that g and h may not be Unitary in general since Lorentz group is noncompact. Thus we write h^{-1} instead of $h^\dagger$.

As a consequences of this analysis, we can use ω_P and ω_Z to construct invariant Lagrangian. The 1-form ω_P is particularly useful, because the coordinate 1-form $\mathrm{d}x^\mu$ is no longer a covariant quantity under broken symmetry transformations. Then ω_P can be used as a substitute of $\mathrm{d}x^\mu$ to construct invariant measure. That is, instead of d^4x, we use $\mathrm{d}\mu = \omega_P^1 \wedge \omega_P^2 \wedge \omega_P^3 \wedge \omega_P^4$ to construct the action, together with an invariant Lagrangian. We can also write $\omega_P^\mu = e^\mu{}_\nu \mathrm{d}x^\nu$ and regard $e^\mu{}_\nu$ as a sort of vierbein. Then the invariant measure is given by $e\mathrm{d}^4x$ where $e \equiv \det(e^\mu{}_\nu)$.

More explicitly, we note that the covariant derivative of Goldstone fields, as in the case of internal symmetry in last section, can be extracted from ω_A, while for any field ψ transforms linearly under the unbroken symmetry, the covariant derivative $\mathrm{D}\psi$ can be constructed as

$$\mathrm{D}\psi = \Big[\mathrm{d} + \omega_V^i D(V_i) + \frac{1}{2}\omega_J^{\mu\nu} D(J_{\mu\nu})\Big]\psi. \tag{26}$$

However, we note again that the Lorentz components of this covariant derivative should be decomposed in terms of ω_P^μ rather than $\mathrm{d}x^\mu$. That is, instead of "$\mathrm{D}\psi = \mathrm{d}x^\mu \mathrm{D}_\mu\psi$", we should write $\mathrm{D}\psi = \omega_P^\mu \mathrm{D}_\mu\psi$. We will make this point more transparent by considering a specific example in the following.

3.2 Inverse Higgs constraint

Before going into examples, however, we should discuss a phenomenon special to broken spacetime symmetries. That is, the number of independent Goldstone fields may not be equal to the number of broken generators. The simple counting rule of Goldstone modes fails when the theory doesn't meet the conditions required by the Goldstone theorem. Basically, the original Goldstone theorem holds only for *global internal* symmetry breaking with the *manifest Poincaré symmetry*. Turning off any of the three conditions could affect the statement of the theorem. For local symmetry, there will be massive gauge bosons rather than massless Goldstone modes, namely the famous Higgs mechanism. For nonrelativistic case, we have the famous examples of phonons from broken translation and rotational symmetry and spin waves from broken $SU(2)$ spin symmetry. Actually there will be no rigorous distinction between spacetime symmetry and internal symmetry in this case. Last, for the spacetime symmetry broken, we have the example of dilaton for broken conformal symmetry as mentioned above. Therefore, to generalize Goldstone's theorem needs a careful study of each of three cases. Here we focus on the case with broken spacetime generators but with manifest Poincaré symmetry.

The physical picture of the mismatch between the number of Goldstone modes and the number of broken generators can be easily illustrated. Suppose the vacuum contains a one-dimensional string lies in a straight line $x = 0$ on the two-dimensional (x, y)-plane. Then the vacuum breaks x-translation and rotation symmetries. However, we will have only one phonon mode instead of two. The reason is clear: we have said that a Goldstone mode can be seen as a localized symmetry transformation generated by broken generators. Here we have two broken generators: the rotation and the x-translation. Now we perform a local rotation to a small interval on the string. But one can equally view this transformation as a local translation in x direction.

From the viewpoint of low energy effective theory, the mismatch comes from the fact that the equations of motion for some of Goldstone fields derived from low energy effective Lagrangian are not dynamical, but are algebraic only. Thus one can eliminate corresponding Goldstone fields by solving the equations of motion. In many cases, however, one can avoid such algebraic equations of motion from the very beginning in coset construction by introducing the *inverse Higgs constraint*.

To be clear, suppose there are two broken generators A^1 and A^2 related to each other by the commutator $[P^\mu, A^1] = \mathrm{i}c^\mu A^2 + \cdots$ where c^μ is a vector-valued constant. Then let's compute a component ω_A^2 of the Maurer-Cartan form. To leading order, the most obvious contribution is from $-\mathrm{i}e^{-\mathrm{i}\pi^2 A^2}e^{-\mathrm{i}x\cdot P}\mathrm{d}e^{\mathrm{i}x\cdot P}e^{\mathrm{i}\pi^2 A^2}$, which gives $\mathrm{d}\pi^2 A^2$. Now the commutator $[P^\mu, A^1] = \mathrm{i}c^\mu A^2 + \cdots$ indicates that there is another piece of contribution from $-\mathrm{i}e^{-\mathrm{i}\pi^1 A^1}e^{-\mathrm{i}x\cdot P}\mathrm{d}e^{\mathrm{i}x\cdot P}e^{\mathrm{i}\pi^1 A^1}$. By using the formula

$$e^{-\mathrm{i}A}Be^{\mathrm{i}A} = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\,\mathrm{ad}\,_A^n B, \tag{27}$$

we see that this term gives $c_\mu \pi^1 \mathrm{d}x^\mu$. Thus we see that

$$\omega_A^2 = (e^\nu{}_\mu \partial_\nu \pi^2 + c_\mu \pi^1 + \cdots)\mathrm{d}x^\mu \tag{28}$$

Now if we set $\omega_A^2 = 0$. Then π^1 can be solved algebraically. This is an example of inverse Higgs constraint. More generally, if two broken generators are related by a commutator involving momentum P^μ in the way described here, one can impose the inverse Higgs constraint to set some components of the Maurer-Cartan form to zero, and then solve for redundant Goldstone parameters. We will illustrate this point explicitly in next subsection. Here we only note that the condition $\omega_A^i = 0$ is a G-invariant constraint since we know that $\omega_A^i \to h\omega_A^i h^{-1}$ under G-transformation.

3.3 Example 1: conformal to Poincaré

As a first application of coset construction for spacetime symmetries, we consider the system with conformal symmetry spontaneously broken to Poincaré symmetry. The conformal symmetry in $(3+1)$-dimensional spacetime is described by $SO(4,2)$. The nonvanishing commutators of the corresponding algebra are,

$$[J^{\mu\nu}, J^{\rho\sigma}] = -\,\mathrm{i}(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\nu\sigma}J^{\mu\rho}), \tag{29}$$

$$[J^{\mu\nu}, P^\lambda] = -\,\mathrm{i}(\eta^{\mu\lambda}P^\nu - \eta^{\nu\lambda}P^\mu), \tag{30}$$

$$[J^{\mu\nu}, K^\lambda] = -\,\mathrm{i}(\eta^{\mu\lambda}K^\nu - \eta^{\nu\lambda}K^\mu), \tag{31}$$

$$[D, P^\mu] = \mathrm{i}P^\mu, \tag{32}$$

$$[D, K^\mu] = -\,\mathrm{i}K^\mu, \tag{33}$$

$$[K^\mu, P^\nu] = -\,2\mathrm{i}(J^{\mu\nu} - \eta^{\mu\nu}D), \tag{34}$$

where D is the generator of dilatation and K^μ are generators of four special conformal transformations. These five generators are all broken generators. From $[K, P] \sim D$, we see that it is possible to introduce the inverse Higgs constraint $\omega_D = 0$. But before that, we should first compute the Maurer-Cartan form. Here we parameterize the coset space by $U(\pi, \tilde{\pi}, x) = e^{\mathrm{i}x^\mu P_\mu}e^{\mathrm{i}\pi D}e^{\mathrm{i}\tilde{\pi}^\mu K_\mu}$. One may also consider other parameterization such as $e^{\mathrm{i}x^\mu P_\mu}e^{\mathrm{i}(\pi D + \tilde{\pi}^\mu K_\mu)}$. This is merely a matter of gauge choice. Now, under our parameterization, the Maurer-Cartan form can be computed to be

$$\omega = -\mathrm{i}U^{-1}\mathrm{d}U = \omega_P^\mu P_\mu + \frac{1}{2}\omega_J^{\mu\nu}J_{\mu\nu} + \omega_D D + \omega_K^\mu K_\mu, \tag{35}$$

where

$$\omega_P^\mu = e^\pi \mathrm{d}x^\mu, \tag{36}$$

$$\omega_J^{\mu\nu} = -2e^\pi(\tilde\pi^\mu \mathrm{d}x^\nu - \tilde\pi^\nu \mathrm{d}x^\mu), \tag{37}$$

$$\omega_D = \mathrm{d}\pi + 2e^\pi\tilde\pi^\mu \mathrm{d}x_\mu, \tag{38}$$

$$\omega_K^\mu = \mathrm{d}\tilde\pi^\mu + \tilde\pi^\mu\mathrm{d}\pi + e^\pi(2\tilde\pi^\mu\tilde\pi_\lambda - \tilde\pi^2\delta_\lambda^\mu)\mathrm{d}x^\lambda. \tag{39}$$

Now it is obvious that we can introduce the inverse Higgs constraint $\omega_D = 0$, which gives

$$\tilde\pi^\mu = -\frac{1}{2}e^{-\pi}\partial^\mu\pi. \tag{40}$$

Then we can use this constraint to eliminate $\tilde\pi^\mu$ from the Maurer-Cartan form, and the resulted low energy theory contains only one Goldstone mode, namely the dilaton field π. For any other Lorentz tensor Φ lying in representation D of Lorentz group, we can construct its covariant derivative via

$$\omega_P^\mu \mathrm{D}_\mu\Phi = \Big[\mathrm{d} + \frac{\mathrm{i}}{2}\omega_J^{\mu\nu}D(J_{\mu\nu})\Big]\Phi. \tag{41}$$

It is ready now to construct the conformal invariant low energy Lagrangian. At lowest order in dilaton's derivative, we have

$$S_0 = M_0^4\int \omega_P^0\wedge\omega_P^1\wedge\omega_P^2\wedge\omega_P^3 = M_0^4\int \mathrm{d}^4x\, e^{4\pi}. \tag{42}$$

At the second order in $\partial_\mu\pi$, we can make use of ω_K^μ to construct the kinetic term,

$$S_2 = M^2\int \omega_K^0\wedge\omega_P^1\wedge\omega_P^2\wedge\omega_P^3 = \frac{M^2}{2}\int \mathrm{d}^4x\, e^{2\pi}(\partial_\mu\pi)^2. \tag{43}$$

At the fourth order, one can consider the form $\omega_K^0\wedge\omega_K^1\wedge\omega_P^2\wedge\omega_P^4$, and also $\eta_{\mu\nu}\omega_K^\mu\wedge(\star\omega_K^\nu)$.

In practice, it is more convenient to construct the Lagrangian directly from the "metric" $g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta\eta_{\alpha\beta} = e^{2\pi}\eta_{\mu\nu}$. One can construct all "diffeomorphism" invariant quantities from this metric as in general relativity. For instance, one can find in this case the Ricci tensor $R_{\mu\nu}$ to be

$$R_{\mu\nu} = 2\partial_\mu\pi\partial_\nu\pi - 2\partial_\mu\partial_\nu\pi - \Box\pi\eta_{\mu\nu} - 2(\partial\pi)^2\eta_{\mu\nu}, \tag{44}$$

and the curvature scalar $R = -6((\partial\pi)^2 + \Box\pi)$. Then the Lagrangian is simply given by operators like $\sqrt{-g}R$, $\sqrt{-g}R^2$, $\sqrt{-g}R_{\mu\nu}R^{\mu\nu}$, etc. Note that the indices here are lowered and raised by $g_{\mu\nu}$ and its inverse. This resembles very much the formulation of general relativity. However, we note that the physical degree of freedom is only the spin-0 dilaton, thus is of course not a gravitational theory. The apparent diffeomorphism invariance of the theory is simply a nonlinear realization of conformal symmetry. Indeed, this is not quite the full diffeomorphism invariance since the form of the metric tells us that we are actually dealing with "conformal flat" theory only.

3.4 Example 2: general relativity

Maybe the most interesting application of the coset construction on global spacetime symmetry is to consider the breaking pattern of affine group to Poincaré group, because the corresponding low energy theory at leading order has precisely the form of general relativity.

The affine symmetry group is a semidirect product of the general linear group $GL(4,\mathbb{R})$ and the translation group $\mathbb{R}^4$. The nonzero commutators of the corresponding algebra are given as follows,

$$
\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= -\,\mathrm{i}(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\nu\sigma}J^{\mu\rho}), && (45)\\
[J^{\mu\nu}, T^{\rho\sigma}] &= -\,\mathrm{i}(\eta^{\mu\rho}T^{\nu\sigma} + \eta^{\mu\sigma}T^{\nu\rho} - \eta^{\nu\rho}T^{\mu\sigma} - \eta^{\nu\sigma}T^{\mu\rho}), && (46)\\
[T^{\mu\nu}, T^{\rho\sigma}] &= \mathrm{i}(\eta^{\mu\rho}J^{\nu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} + \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\nu\sigma}J^{\mu\rho}), && (47)\\
[J^{\mu\nu}, P^{\lambda}] &= -\,\mathrm{i}(\eta^{\mu\lambda}P^{\nu} - \eta^{\nu\lambda}P^{\mu}), && (48)\\
[T^{\mu\nu}, P^{\lambda}] &= -\,\mathrm{i}(\eta^{\mu\lambda}P^{\nu} - \eta^{\nu\lambda}P^{\mu}), && (49)
\end{aligned}
$$

where all T's are broken generators. Treating momenta P's in equal footing with T's, we can parameterize the coset space as usual, by $U(\pi,x) = e^{\mathrm{i}x\cdot P}e^{\mathrm{i}\pi\cdot T}$, where the inner product $\pi\cdot T \equiv \frac{1}{2}\pi^{\mu\nu}T_{\mu\nu}$. Then, with the commutation relations listed above, we can derive the Maurer-Cartan form $\omega = -\mathrm{i}U\mathrm{d}U$, to be

$$
\omega = \omega_P^{\mu}P_\nu + \frac{1}{2}\omega_J^{\mu\nu}J_{\mu\nu} + \frac{1}{2}\omega_T^{\mu\nu}T_{\mu\nu}, \tag{50}
$$

with

$$
\omega_P^{\mu} = e_\nu{}^{\mu}\mathrm{d}x^\nu, \qquad \omega_J^{\mu\nu} = -\frac{1}{2}[e^{-1},\mathrm{d}e]^{\mu\nu}, \qquad \omega_T^{\mu\nu} = -\frac{1}{2}\{e^{-1},\mathrm{d}e\}^{\mu\nu}. \tag{51}
$$

where $e_{\mu\nu} \equiv \big[\exp(\mathrm{i}\pi\cdot D_V(T))\big]_{\mu\nu}$, and D_V denotes vector representation. More explicitly,

$$
D_V(J^{\mu\nu})_{\alpha\beta} = \mathrm{i}(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha), \qquad D_V(T^{\mu\nu})_{\alpha\beta} = \mathrm{i}(\delta^\mu_\alpha\delta^\nu_\beta + \delta^\mu_\beta\delta^\nu_\alpha).
$$

Therefore ω_P^μ provides the needed vierbein 1-form, $\omega_T^{\mu\nu}$ gives the covariant derivative for Goldstone fields, by

$$
\mathrm{D}_\lambda\pi_{\mu\nu}(x) = -\frac{1}{2}(e^{-1})_\lambda{}^\rho\{e^{-1},\partial_\rho e\}_{\mu\nu}, \tag{52}
$$

and $\omega_J^{\mu\nu}$ provides the gauge connection to form covariant derivative for any field Φ transforms linearly under Lorentz group:

$$
\mathrm{D}_\lambda\Phi(x) = (e^{-1})_\lambda{}^\rho\partial_\rho\Phi + \frac{\mathrm{i}}{2}\omega_{\lambda\mu\nu}\Phi, \tag{53}
$$

where $\omega_{\lambda\mu\nu}$ is defined via $(\omega_J)_{\mu\nu} = \omega_P^\lambda\omega_{\lambda\mu\nu}$ and can be solved to be $\omega_{\lambda\mu\nu} = -\frac{1}{2}(e^{-1})_\lambda{}^\rho[e^{-1},\partial_\rho e]_{\mu\nu}$. However, the choice of this gauge connection is not unique; any quantity transforming corrected and ensuring the covariance of derivative of Φ can do the job. For the reason that will be clear in the following, it is particularly instructive to redefine the gauge connection, by

$$
\begin{aligned}
\omega_{\lambda\mu\nu} =& -\frac{1}{2}(e^{-1})_\lambda{}^\rho[e^{-1},\partial_\rho e]_{\mu\nu} - \mathrm{D}_\mu\pi_{\nu\lambda} + \mathrm{D}_\nu\pi_{\mu\lambda}\\
=& -\frac{1}{2}(e^{-1})_\lambda{}^\rho[e^{-1},\partial_\rho e]_{\mu\nu} + \frac{1}{2}(e^{-1})_\mu{}^\rho\{e^{-1},\partial_\rho e\}_{\nu\lambda} - \frac{1}{2}(e^{-1})_\nu{}^\rho\{e^{-1},\partial_\rho e\}_{\mu\lambda}.
\end{aligned} \tag{54}
$$

Now we are ready to construct the effective Lagrangian with covariant quantities like ω_P^μ, $\mathrm{D}_\lambda\pi_{\mu\nu}$, $\mathrm{D}_\mu\Phi$, etc. However, an alternative way of constructing the Lagrangian is to begin from the covariant quantities with manifest geometric meanings. That is, we begin from any Lorentz covariant tensor, and augment it to be a $GL(4,\mathbb{R})$-covariant quantity, by contracting each Lorentz indices with one vierbein factor. For instance, for a Lorentz vector V^α, we can form a $GL(4,\mathbb{R})$-vector V^μ through $V^\mu = (e^{-1})^\mu{}_\alpha V^\alpha$. In particular, we have a generally covariant metric tensor $g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta\eta_{\alpha\beta}$.

4 Incorporating Quantum Anomalies

5 Coset Construction in Nonrelativistic Theories

References