Symmetries of Relativistic Quantum Theories

Zhong-Zhi Xianyu*
Center for High Energy Physics, Tsinghua University, Beijing, 100084

June 24, 2011

Abstract

In this note we review the well-known Coleman-Mandula theorem, concerning the possible symmetries of S matrix in a relativistic quantum theory. The original proof of Coleman and Mandula is presented. Several possible extensions of the theorem by relaxing conditions of Coleman-Mandula theorem are also discussed, including the supersymmetric generalization.

1 Introduction

It ia an interesting question that what kind of symmetries can be adopted by a relativistic quantum theory. By relativistic theory we mean the symmetry group should contain (at least locally) the Poincaré group as its subgroup. Thus the question can be phrased in another way: what is the most general form of symmetry groups of a quantum theory that locally contain Poincaré group as a subgroup? In a variety of examples we met, the symmetries of the theory can be separated into two classes: one described by Poincaré group which we call space-time symmetries, the other commuting with all Poincaré generators, which we call internal symmetries. Therefore the point of the previous question is that whether it is possible or not for a relativistic quantum theory to adopt a symmetry being simultaneously non-space-time and non-internal. Were this the case, loosely speaking, the space-time symmetries and internal symmetries will get mixed. Mathematically, such a non-space-time and non-internal symmetry generator would be outside the Poincaré algebra but have nonzero commutators with Poincaré generators. As a consequence, the generator of such a symmetry will be tensor-like. So another way of asking the question is, if a relativistic theory can admit a tensor-like charge besides the ones in Poincaré algebra?

The Coleman-Mandula (CM) theorem solves the problem with a negative answer, under some quite technical assumptions. The claim of the theorem is that, the symmetry group of a relativistic quantum theory must be a locally direct product of the Poincaré group and internal symmetry groups. In the following, we will try to make this statement more precise, with a number of technical details considered.

The formulation of the theorem is based on the scattering theory. Before stating it, let us first set up the formulation and notations. In scattering theory, the Hilbert space is the Fock space

$$\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)} \oplus \mathscr{H}^{(2)} \oplus \cdots , \tag{1}$$

*E-mail: xyzz06@mails.tsinghua.edu.cn

where $\mathscr{H}^{(n)}$ is the subspace of n-particle states. For example, $\mathscr{H}^{(0)}$ denotes the vacuum, and $\mathscr{H}^{(1)}$ is the space of one-particle states, a basis of which can be chosen to be labeled by $|p, n\rangle$, with p the momentum eigenvalue, n the discrete labels, including spin and particle type.

The S matrix is a unitary operator in $\mathscr{H}$, and a unitary[1] operator U is said to be a symmetry transformation of the S matrix, if (1) U turns one-particle states into one particles states, and acts on multi-particle states as if they were tensor products of one-particle states; (2) U commute with S: $[U, S] = 0$. The T matrix is defined in the usual way, as

$$S = 1 - \mathrm{i}(2\pi)^4\delta(p - p')T. \tag{2}$$

All symmetries of the S matrix form a group. We study the continuous symmetry only, and focus solely on the path-connected component. Then symmetries of S matrix are generated locally by an algebra, consisting of symmetry generators. The CM theorem is a claim on local properties of symmetry group, thus it is enough for us to study the symmetry algebra only. As emphasized in [1], the theorem does not assume the symmetry group to be a finite dimensional Lie group. That is, it can be *a priori* infinitely dimensional, because the particle-finiteness assumption (see below) automatically guarantees the resulted internal symmetry group to be a finite dimensional Lie group.

As stated above, symmetry transformations are represented by unitary operators, thus the corresponding generators are hermitian. Of course these hermitian generators can be defined on the entire Hilbert space, but for our purpose, it is enough to define the generators on one-particle and two-particle Hilbert spaces only, since we only use them in the study of 2-2 scattering processes.

Mathematically, we use $\mathscr{D}$ to denote the subspace of $\mathscr{H}^{(1)}$ consisting of one-particle states $|\Psi\rangle$ with the momentum space wave function $\langle p, j, n|\Psi\rangle$ being test function of p. By test function we mean a infinitely differentiable function with compact support. Then, let $|\Psi_1, \Psi_2\rangle \equiv |\Psi_1\rangle \otimes |\Psi_2\rangle$ and $|\Phi_1, \Phi_2\rangle \equiv |\Phi_1\rangle \otimes |\Phi_2\rangle$ be two two-particle states in $\mathscr{D} \otimes \mathscr{D}$. Now, we assume that any symmetry transformation in a small neighborhood of the identity is contained in an one-parameter group described by real parameter t, that is to say, any such unitary operators can be written as a function of t, $U = U(t)$. Then, we can use it do define the corresponding symmetry generator in two-particle space $\mathscr{H}^{(2)}$ to be any distribution satisfying

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\langle\Phi_1, \Phi_2|g(t)|\Psi_1, \Psi_2\rangle\Big] = \langle\Phi_1, \Phi_2|A|\Psi_1, \Psi_2\rangle = \langle\Phi_1|A|\Psi_1\rangle\langle\Phi_2|\Psi_2\rangle + \langle\Phi_1|\Psi_1\rangle\langle\Phi_2|A|\Psi_2\rangle, \tag{3}$$

with the requirement that symmetry generators should commute with the S,

$$\langle\Phi_1, \Phi_2|S^\dagger AS|\Psi_1, \Psi_2\rangle = \langle\Phi_1, \Phi_2|A|\Psi_1, \Psi_2\rangle. \tag{4}$$

We denote the space of all generators satisfying the two equations above by $\mathscr{A}$. The matrix elements of A between two one-particle momentum eigenstates, $\langle p', n'|A|p, n\rangle$ will be denoted by $A_{n',n}(p', p)$, and discrete indices will often be suppressed in the following.

With the preliminaries above, we are now ready to state the CM theorem in its standard form:

[1] By a well-known theorem of Wigner, elements of a symmetry group are always represented by unitary or antiunitary operators in Hilbert space. Since only continuous symmetries are considered in this note, all symmetry transformations are unitary.

Theorem (Coleman-Mandula) *Let G be a connected symmetry group of the S matrix, and let the following five conditions hold:*

1. *(Lorentz invariance) G contains a subgroup locally isomorphic to Poincaré group P.*
2. *(Particle-finiteness) For any finite M, there are only a finite number of particle types with mass less than M.*
3. *(Weak elastic analyticity) Elastic-scattering amplitudes are analytic functions of center-of-mass energy s and invariant momentum transfer t, in some neighborhood of the physical region, except at normal thresholds.*
4. *(Occurrence of scattering) Let $|p, p'\rangle$ be a two-particle state form by any two one-particle momentum eigenstates. Then $T|p, p'\rangle \neq 0$ except for some isolated values of s.*
5. *(Kernel as distribution) There is a neighborhood of the identity G such that every element of G in this neighborhood lies on some one-parameter group $U(t)$, and for any well-defined one-particle states $|\Phi\rangle$ and $|\Psi\rangle$,*

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|U(t)|\Phi\rangle = \langle\Psi|A|\Phi\rangle \tag{5}$$

defines a well-behaved functional, linear in $|\Phi\rangle$ and antilinear in $\langle\Psi|$.

Then, G is locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

In next section, we give the proof of the theorem following [1].

2 Proof of the CM Theorem

We separate the proof into three parts. The first two parts consist of two lemmas. The first lemma shows that matrix elements of symmetry generators between one-particle states do not vanish only when the two one-particle states share the same momentum; the second lemma is a partial statement of the CM theorem for generators commuting with momentum operator. Then in the third part, we generalize the statement of Lemma 2 to the entire space of symmetry generators, with the aid of Lemma 1.

2.1 Part 1: Symmetry generators are diagonal in momentum

Lemma 1. *$A(p', p) \equiv \langle p'|A|p\rangle$ vanishes constantly when $p' \neq p$.*

In particular, this lemma implies that a symmetry generator A cannot turn a one-particle state into another one with different mass.

Proof of Lemma 1. Let f be a test function in momentum space and $\tilde{f}$ its Fourier transform, with the support R assumed not to contain zero. We use f, together with a given symmetry generator A, to construct another operator A_f:

$$A_f \equiv \int \mathrm{d}^4 a\, U^\dagger(1, a) A U(1, a)\tilde{f}(a). \tag{6}$$

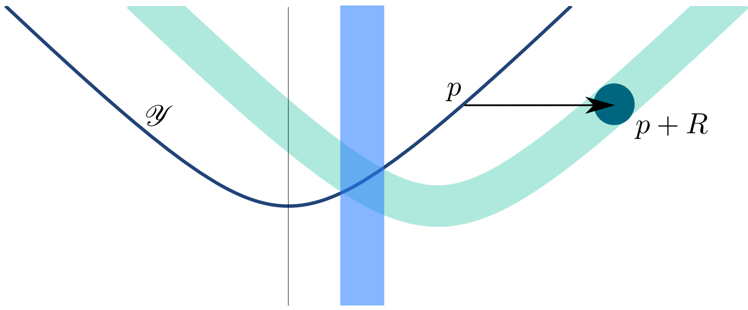

Figure 1: An illustration of the action of A_f on a given mass shell. The support R of the test function f is chosen to lie in the a small disk on the positive x axis. Then for every point p on the hyperboloid, only points in the small disk $p + R$ has nonvanishing matrix elements of A_f together with p. As p moves in the hyperboloid, $p + R$ sweeps over a bold hyperboloid that intersects the original hyperboloid within the range marked by the vertical blue strip. Then the original mass shell except the part covered by this blue strip is the region $\mathscr{Y}$.

It's straightforward to show that operator A_f is also a symmetry generator of the S matrix That is, A_f satisfies (5). Then, using the property of one-particle momentum eigenstate

$$U(1, a)|p\rangle = e^{-\mathrm{i}p\cdot a}|p\rangle,$$

we have

$$\begin{aligned} A_f(p', p) = \langle p'|A_f|p\rangle &= \int \mathrm{d}^4 a\, \tilde{f}(a)\langle p'|U^\dagger(1, a)AU(1, a)|p\rangle \\ &= \int \mathrm{d}^4 a\, \tilde{f}(a)e^{-\mathrm{i}(p-p')\cdot a}\langle p'|A|p\rangle = f(p - p')A(p', p), \end{aligned} \tag{7}$$

which shows that the matrix element of A_f between two one-particle states is nonzero, only when the momentum difference $p' - p$ is in the support R of f.

By the particle-finiteness assumption, the support of one-particle states in momentum space is restricted to countable mass shells. Then, it is easy to show the following statement to be true: there is a region $\mathscr{Y}$ in each mass shell such that every point in $\mathscr{Y}$ when added by any point in R, will be outside any of the mass shells, provided that R is sufficiently small and does not contain zero. As a consequence, any one-particle state with support contained in $\mathscr{Y}$ will be annihilate by A_f. This can be clarified by careful mathematical constructions, which we do not bother to spell out explicitly. Instead, we illustrate the meaning of this statement in Fig. 1.

Now let's consider a 2-2 scattering process with initial momenta p, q and final momenta p', q' chosen such that p is out of $\mathscr{Y}$ while q, p' and q' are in $\mathscr{Y}$, with momentum conservation $p + q = p' + q'$ satisfied. Then we see that A_f must annihilate the one-particle states associated with q, p' and q' but not p. Then this scattering process is forbidden by the conservation law associated with A_f and thus has vanishing S matrix element. In the rest frame of momentum p, any rotation of the system won't change this result, but will change the momenta q, p' and q' continuously. This implies that the scattering amplitude vanishes at least for a continuous interval of center-of-mass energy $s = (p + q)^2$, which contradicts the scattering assumption that demands the amplitudes vanish only at isolated values of s. Thus we see that the state associated with p must vanish, and A_f will annihilate all one-particle states lying on the lowest mass shell.

Repeat this argument to the succeeding mass shells, we conclude that $A_f(p',p) = f(p-p')A(p',p) = 0$. Now that f is arbitrarily chosen except that its support does not contain zero, we see that $A(p',p)$ must be zero except when $p'=p$. This finish the proof of Lemma 1.

We note that by the distribution assumption for the kernel of symmetry generators, any distribution in $\mathscr{A}$ is a polynomials in delta functions $\delta(p-p')$ and its finite-order derivatives $\frac{\partial}{\partial p_{\mu_1}}\cdots\frac{\partial}{\partial p_{\mu_n}}\delta(p-p')$. For the generator $A(p',p)$ that acts only on a mass shell, the derivatives must be tangent to the mass shell, thus $A(p',p)$ must be a polynomial of tangent derivative ∇_μ given by

$$\nabla_\mu = \frac{\partial}{\partial p^\mu} - \frac{p_\mu p_\nu}{m^2}\frac{\partial}{\partial p_\nu}.$$

Then it's straightforward to check that the commutator between A and squared momentum $P^\mu P_\mu$ vanishes,

$$[A, P^\mu P_\mu] = 0, \tag{8}$$

which is a mathematical way to say that A does not affect the mass of an one-particle state.

2.2 Part 2: The subalgebra commuting with momentum is internal

The second part of the proof concerning the symmetry generators which are commuting with space-time translation. Let $\mathscr{B}$ be the space of symmetry generators that commute with space-time translations. Then the main proposition we will to prove in this part is the following:

Lemma 2. *Any element in $\mathscr{B}$ has the form:*

$$B = a^\mu P_\mu + b, \tag{9}$$

where a^μ is constant four-vector, and b is a constant, being space-time scalar and Hermitian matrix in Hilbert space.

Before proving this lemma, we at first make some remarks. Particle states form representations of symmetry generators. That is to say, for instance, for any $B \in \mathscr{B}$ and any one-particle state $|p,n\rangle$,

$$B|pn\rangle = \sum_m b_{mn}(p)|pm\rangle, \tag{10}$$

where $b_{mn}(p)$ is a hermitian matrix. For two-particle states $|p_1n_1, p_2n_2\rangle$, then, the action of B is constructed from the action on corresponding one-particle states:

$$\begin{aligned} B|p_1n_1,p_2n_2\rangle &= \sum_{n_1'} b_{n_1'n_1}(p_1)|p_1n_1',p_2n_2\rangle + \sum_{n_2'} b_{n_2'n_2}(p_2)|p_1n_1,p_2n_2'\rangle \\ &= \sum_{n_1',n_2'} b_{n_1'n_2',n_1n_2}(p_1,p_2)|p_1n_1',p_2n_2'\rangle, \end{aligned} \tag{11}$$

where $b_{n_1'n_2',n_1n_2}(p_1,p_2)$ is defined by

$$b_{n_1'n_2',n_1n_2}(p_1,p_2) = b_{n_1'n_1}(p_1)\delta_{n_2'n_2} + b_{n_2'n_2}(p_2)\delta_{n_1'n_1}. \tag{12}$$

The action of B on general multiparticle states can be determined in a similar way. Furthermore, the fact that B is a symmetry of the S matrix, i.e., $BS = SB$ gives that in the two-particle space,

$$b(q_1, q_2)S(q_1, q_2, p_1, p_2) = S(q_1, q_2, p_1, p_2)b(p_1, p_2), \tag{13}$$

where the discrete indices have been suppressed.

For convenience, note that the Assumption 4 (occurrence of scattering) says that $T|p, q\rangle \neq 0$ except for isolated values in s, we will call the momenta (p, q) associated with these isolated s values a *null pair.*

Now we begin the proof.

Proof of Lemma 2. Let's separate $B \in \mathscr{B}$ into a multiple of identity (pure trace) and a traceless part $B \equiv \operatorname{tr}(B) + B^\sharp$, for the reason that will be clear in the following. Then the first step of our proof is to show that the representation of $B^\sharp \in \mathscr{B}$ with two-particle states[2] on a given mass shell is a loyal representation, i.e., the homomorphic mapping $B^\sharp \mapsto b^\sharp(p_1, p_2)$ *for fixed* p_1, p_2 is in fact an isomorphism. This can be achieved by showing that the vanishing of a combination of representation matrices $\sum c_\alpha b^\sharp_\alpha(p_1, p_2) = 0$ *for fixed* (p_1, p_2) leads to the vanishing of the corresponding linear combination of generators, $\sum c_\alpha B^\sharp_\alpha = 0$. This is true if we can show that $\sum c_\alpha b^\sharp_\alpha(p_1, p_2) = 0$ *for fixed* (p_1, p_2) leads to $\sum c_\alpha b^\sharp_\alpha(q) = 0$ *for all* p, since the representation of $B^\sharp$ on the entire Fock space can be generated by its representation on one-particle space (as shown in (12) for two-particle case), then the vanishing of $\sum c_\alpha b^\sharp_\alpha(p)$ for all p assures that $\sum c_\alpha B^\sharp_\alpha = 0$.

Now, from the (13), and the fact that $S(q_1, q_2, p_1, p_2)$ is non-singular for non-null pairs (q_1, q_2) and (p_1, p_2) satisfying $q_1 + q_2 = p_1 + p_2$, we see that the condition $\sum c_\alpha b^\sharp_\alpha(p_1, p_2) = 0$ *for fixed* (p_1, p_2) yields $\sum c_\alpha b^\sharp_\alpha(q_1, q_2) = 0$ *for* (q_1, q_2) *subjected to the condition* $q_1 + q_2 = p_1 + p_2$, which in turn leads to $\sum c_\alpha b^\sharp_\alpha(q_1) = \sum c_\alpha b^\sharp_\alpha(q_2) = 0$, since $b^\sharp$ is traceless[3]. This is not enough though, since $\sum c_\alpha b^\sharp_\alpha(q_1) = 0$ only holds for those q_1 satisfying $q_1 = p_1 + p_2 - q_2$. Since p_1 and p_2 are fixed, there is only one DOF (namely q_2) for q_1, not enough to cover the whole mass shell. To enlarge the DOFs, we note that $\sum c_\alpha b^\sharp_\alpha(p_1, p_2)$ already implies that $\sum c_\alpha b^\sharp_\alpha(p_1) = 0$. Together with $\sum c_\alpha b^\sharp_\alpha(q_1)$, we get $\sum c_\alpha b^\sharp_\alpha(p_1, q_1) = 0$. Then the original condition that $\sum c_\alpha b^\sharp_\alpha(p_1, p_2)$ for both momenta p_1 and p_2 fixed has been relaxed to the same equality but with only one momenta fixed. This releases an additional DOF, so that $\sum c_\alpha b^\sharp_\alpha(q_1) = 0$ holds for q_1 varying on a continuous 2-dimensional range on the mass shell. Thus we reach the partial conclusion that $\sum c_\alpha b^\sharp_\alpha(p) = 0$ constantly on the mass shell, namely, the mapping from generators $B^\sharp$ to hermitian matrices $b^\sharp(p_1, p_2)$ for fixed p_1 and p_2 is indeed an isomorphism. This finishes the first part of our proof.

The second part of the proof is to show that all traceless generators $B^\sharp$ must be internal, and form a *finite dimensional Lie algebra.*

The argument begins with the note that all traceless hermitian matrices $b(p_1, p_2)$ *for fixed momenta*, form a closed algebra $\mathfrak{B}$, homomorphic to the algebra of $SU(N) \otimes SU(N)$, with N the number of particle types of given mass. Thus $\mathfrak{B}$ must be a direct sum of a compact semisimple

[2]We choose the two-particle states because we want to base our argument on the 2-2 scattering process, where (all-most non-singularity) assumption of S-matrix in the condition of the theorem apply.

[3]This is the place where the tracelessness of $B^\sharp$ works. Note that if we begin the proof with B directly (without its trace subtracted), then with (12), the vanishing of $\sum c_\alpha b_\alpha(q_1, q_2)$ can not yield the vanishing of $\sum c_\alpha b_\alpha(q)$, but give $\sum c_\alpha b_\alpha(q)$ is proportional to identity and odd in momentum q only.

Lie algebra and an Abelian Lie algebra. In the next a few paragraphs, we show that all generators $B^\sharp$ commute with all Lorentz generators, for $B^\sharp$ in the semisimple Lie algebra and in the Abelian Lie algebra, respectively.

For the case of semisimple algebra, an arbitrary generator $B^\sharp_\alpha$ transforms under the Lorentz transformation $U(\Lambda)$ as

$$U(\Lambda)B^\sharp_\alpha U^{-1}(\Lambda) = \sum_\beta D^\beta{}_\alpha(\Lambda)B^\sharp_\beta, \tag{14}$$

where $D^\beta{}_\alpha(\Lambda)$ furnish a finite-dimensional representation of Lorentz group. Let's show that this representation can be made into a unitary representation. In fact, from the transformation of structure constant of this semisimple algebra,

$$C^\gamma_{\alpha\beta} = D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)D^{\gamma'}{}_\gamma(\Lambda^{-1})C^{\gamma'}_{\alpha'\beta'},$$

we can find

$$g_{\alpha\beta} = D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)g_{\alpha'\beta'},$$

where $g_{\alpha\beta} \equiv C^\gamma_{\delta\alpha}C^\delta_{\gamma\beta}$ is the algebra metric. Since the algebra is semisimple, this metric is positive-definite. Thus $g^{-1/2}D(\Lambda)g^{-1/2}$ indeed form a finite-dimensional unitary representation of Lorentz group. However, since Lorentz group is non-compact, thus the only finite-dimensional representation is the trivial one. That is, all $D(\Lambda) = 1$. As a consequence, we see that $B^\sharp_\alpha$ commutes with all Lorentz generators.

Then consider the Abelian algebra. In this case, consider any non-null pair (p, q) and go to the frame in which p and q are aligned in z-direction. Then the commutator $[P, [J, B^\sharp_\alpha]] = 0$ by Jacobi identity:

$$[P, [J, B^\sharp_\alpha]] + [J, [B^\sharp_\alpha, P]] + [B^\sharp_\alpha, [P, J]] = 0,$$

since we know that $[B^\sharp_\alpha, P] = 0$, and $[J, P] \propto P$, thus the last two terms on the l.h.s. vanish. That is to say, $[J, B^\sharp_\alpha]$ must be in $\mathscr{B}$. On the other hand, as an abelian generator, $B^\sharp_\alpha$ commutes with all elements in $\mathscr{B}$. Thus, we further have $[B^\sharp_\alpha, [J, B^\sharp_\alpha]] = 0$. Now, consider the the matrix element of this commutator between two-particle states $|m, n\rangle$, on which

$$J|m, n\rangle = \sigma(m, n)|m, n\rangle, \qquad B^\sharp_\alpha|m, n\rangle = (b^\sharp_\alpha)_{mn}|m, n\rangle,$$

where m, n denote the particle species on a given mass shell and run over a finite discrete range, we get

$$0 = \langle m, n|[B^\sharp_\alpha, [J, B^\sharp_\alpha]]|m, n\rangle = \sum_{m',n'} \Big(\sigma(m', n') - \sigma(m, n)\Big)\big|(b^\sharp_\alpha)_{mn,m'n'}\big|^2. \tag{15}$$

Suppose that $\sigma(m, n)$ does not share a same value σ, then there exists a pair (m, n), such that $\sigma(m, n)$ is the smallest one. Then the r.h.s. of the expression above is positive-definite, contradicting the equality, unless all $(b^\sharp_\alpha)_{mn,m'n'}$ vanishes. Thus we conclude that $(b^\sharp_\alpha)_{mn,m'n'}$ must vanish when $\sigma(m, n) \neq \sigma(m', n')$, which is equivalent to the fact that $[J, B^\sharp_\alpha] = 0$. Now we can arbitrarily choose the non-null pair (p, q) to show that $[J_{\mu\nu}, B^\sharp_\alpha] = 0$.

Now we have demonstrate that any traceless symmetry generator $B^\sharp_\alpha$ commuting with momentum operator also commutes with Lorentz generator, thus is internal.

Finally let us deal with the trace part $\mathrm{tr}\,(B)$. From (13), we see immediately that $\mathrm{tr}\,b(q_1,q_2) = \mathrm{tr}\,b(p_1,p_2)$. Then by (12), we have further

$$\mathrm{tr}\,b(q_1) + \mathrm{tr}\,b(q_2) = \mathrm{tr}\,b(p_1) + \mathrm{tr}\,b(p_2), \tag{16}$$

for any q_1, q_2 satisfying $q_1 + q_2 = p_1 + p_2$. That is to say, $\mathrm{tr}\,b$ must be a linear function in p, and $\mathrm{tr}\,B$, in turn, must be a multiple of the momentum operator P_μ,

$$\mathrm{tr}\,B = a^\mu P_\mu, \tag{17}$$

and this finish the proof of Lemma 2.

2.3 Proof of the theorem

Now we are in the position to prove the theorem, with the two lemmas stated and proved above. Firstly, as we have noted in the proof of Lemma 1, any generator $A \in \mathscr{A}$ has the following form:

$$A = \sum_{n=0}^{N} A_{\mu_1\cdots\mu_n}(p)\frac{\partial}{\partial p_{\mu_1}}\cdots\frac{\partial}{\partial p_{\mu_n}},$$

by distribution assumption (Assumption 5). If $N > 0$, then the N-fold commutators between A and P_μ can be easily evaluated to be

$$[P_{\mu_1},[P_{\mu_2},\cdots[P_{\mu_n},A]\cdots]] = A^{(N)}_{\mu_1\cdots\mu_N}(p), \tag{18}$$

which is still in $\mathscr{A}$, and also commutes with P^μ. That is to say, this N-fold commutator is now in $\mathscr{B}$, where Lemma 2 applies. Thus $A^{(N)}_{\mu_1\cdots\mu_N}(p)$ has the form of

$$A^{(N)}_{\mu_1\cdots\mu_N}(p) = a_{\mu\mu_1\cdots\mu_N}P^\mu + b_{\mu_1\cdots\mu_N}.$$

On the other hand, let's replace the outermost element P_{μ_1} in the N-fold commutator by $P_\mu P^\mu$. The result vanishes according to (8):

$$[P_\mu P^\mu,[P_{\mu_2},\cdots[P_{\mu_n},A]\cdots]] = a_{\mu\mu_1\cdots\mu_N}P^\mu P^{\mu_1} + b_{\mu_1\cdots\mu_N}P^{\mu_1} = 0. \tag{19}$$

This result is independent of the momentum on a mass shell, thus the vanishing of the term linear in P^μ implies that $b_{\mu_1\cdots\mu_N} = 0$, and the vanishing of the term quadratic in P^μ implies that $a_{\mu\mu_1\mu_2\cdots\mu_N} = a_{\mu_1\mu\mu_2\cdots\mu_N}$. But a is totally symmetric in its indices, thus we conclude that $a_{\mu\mu_1\mu_2\cdots\mu_N} = 0$ for $N > 1$.

Thus there are only two nontrivial cases, $N = 0$ and $N = 1$. In the latter case, we have

$$A = A^{(1)}_\mu\frac{\partial}{\partial p_\mu} = a_{\nu\mu}P^\nu\frac{\partial}{\partial p_\mu},$$

with $a_{\nu\mu}$ antisymmetric. This is just the generator of Lorentz transformations. While in the case $N = 0$, A does not contain p-derivative, and commutes with P_μ. Lemma 2 applies again, indicating that A must be a momentum operator or a internal symmetry generator, or their linear combinations. In conclusion, the generator A must be a sum of Poincaré generators and internal symmetry generators. In the language of groups, the symmetry of S-matrix in a relativistic theory must be a locally direct product of Poincaré group and a finite dimensional internal symmetry Lie group. This finish the proof of the Colema-Mandula theorem.

3 Remarks

The results of CM theorem is quite impressive, but its assumptions are rather technical. Now we will show that these apparently complicated conditions are in fact quite critical, in the sense that violating or relaxing of one of these conditions will in general breaks the conclusion of the theorem. Let us now examine some of these conditions.

The first assumption, the Poincaré invariance, as we have seen, is very crucial in proving the theorem, which is reflected not only in the spectrum structure of one-particle or multi-particle states, as the infinite dimensional unitary representations of Poincaré group, but also in the key feature of non-compactness of Poincaré group. With this in mind, we states a similar theorem without a proof concerning a relativistic theory with massless particles only. In this case, the space-time symmetry is enlarged to conformal group. In 4 dimensional case, the conformal group is isomorphic to $SO(4,2)$, which contains Poincaré group as a subgroup, and is also noncompact. Thus the "massless" edition of the CM theorem is, the symmetry of S-matrix is described by a locally direct product of conformal group and finite dimensional internal symmetry Lie group.

But the theorem no longer holds if one tries to replace the Poincaré group by Galilean group, since the latter is compact. So there is no non-relativistic edition of CM theorem, and in fact, "counterexamples" exist in a non-relativistic theory. For instance, in the nonrelativistic theory of nucleus, one can combine the space-time rotational symmetry $SU(2)_J$ together with isospin symmetry $SU(2)_I$ into a larger simple group $SU(4)$. Particles form "supermultiplets" under the action of this group. One can also combine the "flavour" $SU(3)$ group of quarks and rotational group $SU(2)_J$ into a larger group $SU(6)$ ([3], see also [2], Chp. 24, App. A).

3.1 Supersymmetric Extension

The CM theorem claimed that all symmetry tensorial generators other than Poincaré algebra are actually space-time scalar. It says nothing about spinorial generators. From spin-statistic relations, that means there are other possibilities besides Lie algebras consisting of commuting generators, that is, the so-called graded Lie algebras, consisting of commuting and anticommuting generators. This possible extension of the CM theorem is given by Haag, Łopuszański and Sohnius (HLS)[5], which states that besides Poincaré generators and scalar internal generator, there can be spinorial anticommuting generators that acts as the supersymmetry of the theory. In addition, the only possible spinorial generators are spin-$\frac{1}{2}$ operators.

References

[1] S. Coleman and J. Mandula: Phys. Rev. **159**, 1251 (1967).

[2] S. Weinberg: *The Quantum Theory of Fields, Vol.3: Supersymmetry*, Cambridge, 2000.

[3] B. Sakita: Phys. Rev. **21**, B1756 (1964).

[4] O. Pelc and L. P. Horwitz: arXiv:hep-th/9605147.

[5] R. Haag, J. Łopuszański and M. Sohnius, *Nucl. Phys. B* **88**, 257 (1975).

[6] J. Wess and J. Bagger, *Supersymmtry and Supergravity*, (2nd ed.) Princeton University Press, 1991.