清 华 大 学

综 合 论 文 训 练

题目：Vilkovisky-DeWitt有效作用量，规范无关的量子引力修正，与所有规范理论的渐进自由

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中文摘要

在能量尺度低于Planck能标的区域内，可以从有效场论的观点出发研究引力的量子效应，即使目前人们尚未找到完整的量子引力理论。随之而来的一个有趣问题是，量子引力效应将如何修改其他三种相互作用的耦合强度随能量尺度的跑动行为。依照传统，可使用有效作用量方法处理此类问题。然而，人们已经发现，在引力理论中使用传统有效作用量方法不能保证物理结果的规范无关性。

为避免传统方法的缺陷，本文采用了Vilkovisky与DeWitt于二十世纪八十年代发展的一套几何有效作用量方法，并以此研究了量子引力效应对标准模型中所有规范相互作用随能量跑动的修正。这套方法可以保证物理结果的规范无关性。

本文介绍了Vilkovisky-DeWitt有效作用量的基本内容，并以Coleman-Weinberg模型为例详细展示了该方法的诸多要点。在此基础上，本文研究了引力效应对标量phi-4理论、$U(1)$规范场论，以及Yang-Mills理论的修正，并在一圈图近似下计算了这三种模型β函数的领头项。我们发现，当考虑进量子引力效应后，在包括Abel与非Abel在内的所有规范理论中都出现了新的紫外固定点，表现出渐进自由。这些紫外固定点的出现提示了在Planck能标附近三种基本规范相互作用的大统一。这个大统一的出现是引力效应的结果，故而不再需要超对称的存在。

关键词：Vilkovisky-DeWitt方法　量子引力　规范相互作用　渐进自由　大统一

ABSTRACT

At the energy below the Planck scale, the quantum properties of gravitation can be well studied under the philosophy of effective field theory, even though a complete quantum theory of gravity is still absent now. Then there arises an interesting question, namely, how the quantum-gravitational corrections will affect the running of the gauge couplings of the other three types of fundamental interactions. Traditionally, these kinds of problems can be approached by effective action method. However, it has been shown that traditional effective action method can not guarantee the gauge invariance of physical results in this case.

In this thesis, we investigate the quantum-gravitational corrections to the running of all gauge couplings in standard model, by using the geometrical effective action method developed by Vilkovisky and DeWitt in 1980s, rather than the traditional one. The gauge invariance of the results can be guarantee manifestly.

The basics of Vilkovisky-DeWitt effective action is introduced with an detailed illustration with Coleman-Weinberg model. Then, we fully study the gravitational corrections to the scalar phi-four interactions, $U(1)$ gauge theory, as well as $SU(N)$ gauge theories. The leading terms of β functions of these theories are calculated at 1-loop level. We find that all gauge theories, both the Abelian and the non-Abelian ones, exhibit asymptotic freedom, when gravitational corrections enter. These ultraviolet fixed points of gauge theories may also imply a new grand unification without the requirement of supersymmetry.

Key words: Vilkovisky-DeWitt method   quantum gravity   gauge interactions   asymptotic freedom   grand unification

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记号与约定

- 时空度规以$g_{\mu\nu}$表示，其中，约定Minkowski时空的度规为：$\eta_{\mu\nu} = (+,-,-,-)$。
- 关于广义相对论中各基本量，我们使用如下的约定

$$\begin{aligned}
&\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} \\
&R = g^{\mu\nu} R_{\mu\nu}; \\
&R_{\kappa\nu} = R^{\lambda}{}_{\kappa\lambda\nu}; \\
&R^{\lambda}{}_{\kappa\mu\nu} = \partial_\nu \Gamma^{\lambda}_{\kappa\mu} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\kappa\mu} - (\mu \leftrightarrow \nu); \\
&\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}).
\end{aligned}$$

- 场空间的点以ϕ_i表示，其中的拉丁指标$i, j, \cdots$代表所有的时空点坐标、Lorentz指标，以及内部对称性指标。
- 在第二章中，希腊指标$\alpha, \beta, \cdots$是规范空间的指标，大写拉丁指标$A, B, \cdots$是轨道空间的指标。
- 场空间的度规以G_{ij}表示，由其诱导的Christoffel联络以Γ^k_{ij}表示，规范理论中完整的场空间联络以$\tilde{\Gamma}^k_{ij}$表示。
- 在处理场空间的章节中，我们使用推广的Einstein求和约定：场空间中重复指标的求和包括Lorentz指标、内部对称性指标的求和，以及时空坐标的积分。
- 指标的对称化：我们约定$A_{(i}B_{j)} \equiv \frac{1}{2}(A_i B_j + A_j B_i)$。

第 1 章 引言

1.1 基本相互作用

自然界中存在四种基本相互作用：万有引力，电磁力，强相互作用与弱相互作用。这些相互作用被称为“基本”，是由于在现有理论中，它们无法再为其它相互作用所描述。“四种基本相互作用”的图景为二十世纪物理学界广泛接受[1]。

在人类探索基本相互作用的征途中，“统一”是一个无可争议的关键词。历史上第一种处理基本相互作用的理论是Newton引力理论。以此理论为基础，并结合三条运动定律，Newton为天空中的星体运行与地面上诸多力学现象提供了统一的物理解释。Newton理论是如此成功，以至于在其后百年之内，人们普遍相信它就是描述外在世界的终极真理。

继Newton之后的下一次突破来自Maxwell。这位十九世纪伟大的物理学家将电现象与磁现象统一进以他的名字命名的方程组中。在此基础上发展起来的电动力学至今仍然是经典场论的典范。

二十世纪初的科学革命为物理学引进了全新的概念。在此时诞生的相对论和量子理论已成为现代物理学的两大基石。广义相对论作为一种处理引力的经典场论，不仅在比照于实验结果时极为成功，其简洁与优美的理论结构也给人们留下了深刻的印象。另一方面，以狭义相对论和量子力学为基础的量子场论已经成功地将强、弱、电磁等三种基本相互作用以规范场论的方式纳入到它的理论框架中。以此为基础形成的粒子物理标准模型取得对了巨大的成功。

与此形成鲜明对照的是，由于来自理论和实验两方面的困难，人们目前尚未找到一个完整的量子引力理论。从理论上，广义相对论结构特殊，对其直接做量子化时存在不可重整等问题，这意味着在极高能量（即Planck能标，$E_{\mathrm{Planck}} \sim 10^{1}9\mathrm{GeV}$）下，广义相对论将失效。换言之，一个完整自洽的量子引力理论并非广义相对论的简单量子化。在实验上，由于引力表现出量子效应的典型能量尺度，即Planck尺度，远远高于目前人类实验所能触及的范围（$10^{3}\mathrm{GeV}$），因此当下不能指望从实验上直接探索量子引力的特性。

尽管我们没有完整的量子引力理论可供使用，但是根据现代有效场论的观

点[2]，无论量子引力的最终形式是什么，在能量尺度低于Planck能标时，用量子场论模型处理引力都是一个好的近似。这是本文的出发点之一。

1.2 引力的场论方法

以场论观点出发，广义相对论可被陈述为如下的作用量：

$$S = S_G + S_\Lambda + S_{\text{matter}}. \tag{1-1}$$

此作用量的前两项描写纯引力的物理规律：

$$S_G = \frac{1}{16\pi G}\int \mathrm{d}^d x\ \sqrt{-g}R = \frac{1}{\kappa^2}\int \mathrm{d}^d x\ \sqrt{-g}R, \tag{1-2}$$

$$S_\Lambda = -\frac{2}{\kappa^2}\int \mathrm{d}^d x\ \sqrt{-g}\Lambda. \tag{1-3}$$

其中, S_G 即著名的Hilbert-Einstein作用量，而S_Λ是宇宙项。另外，G是Newton引力常量，从而 $\kappa = \sqrt{16\pi G}$是小量，我们将用它作为作用量的展开量，Λ是宇宙学常数。根据目前的实验结果，Λ 非零，但极小[3]。

作用量(1-1)中的最后一项中包含物质场，其形式依赖于物质场的具体模型。

我们在此处所给出作用量之形式随着各物理量取法的不同而不同[4]。在关于引力的文献中，这些取法并不统一。故而有必要在此列出本文所用的一套约定：

$$\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} \tag{1-4}$$

$$R = g^{\mu\nu}R_{\mu\nu}; \tag{1-5}$$

$$R_{\kappa\nu} = R^{\lambda}{}_{\kappa\lambda\nu}; \tag{1-6}$$

$$R^{\lambda}{}_{\kappa\mu\nu} = \partial_\nu \Gamma^\lambda_{\kappa\mu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\kappa\mu} - (\mu \leftrightarrow \nu); \tag{1-7}$$

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}). \tag{1-8}$$

另外，我们取平直Minkowski时空的时空度规 $(+,-,-,-)$。

由于我们考虑的是能量并不高（与Planck能标相比）、引力并不很强的情形，故而，我们将在背景度规附近展开Hilbert作用量$\eta_{\mu\nu}$。为此，依下式在完整的度规$g_{\mu\nu}$中分离出微扰量$h_{\mu\nu}$：

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{1-9}$$

由于κ的量纲为[能量]$^{-1}$，而度规$g_{\mu\nu}$本身是无量纲量，故$h_{\mu\nu}$具有能量的量纲。

由此，度规的逆$g^{\mu\nu}$即可按下式展开：

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}_{\alpha} h^{\alpha\nu} + O(\kappa^3). \tag{1-10}$$

在此强调，$h_{\mu\nu}$的指标用平直的背景度规$\eta_{\mu\nu}$上升或下降（而非完整度规$g_{\mu\nu}$）。此外，我们还将使用如下记号表示$h_{\mu\nu}$的迹：

$$h \equiv h^{\mu}_{\mu} = \eta^{\mu\nu} h_{\mu\nu}. \tag{1-11}$$

现在，将以上对度规的展开式代入Hilbert-Einstein作用量，即可得到作用量的展开式。我们将推导置于附录，在此处列出结果。保留到$O(\kappa^0)$，纯引力场部分的作用量为：

$$S_G = \int \mathrm{d}^d x \, \tfrac{1}{4}[h\partial^2 h - h^{\mu\nu}\partial^2 h_{\mu\nu} + 2h_{\mu\lambda}\partial^{\mu}\partial_{\nu}h^{\nu\lambda} - 2h\partial^{\mu}\partial^{\nu}h_{\mu\nu}] \tag{1-12}$$

$$S_\Lambda = -\int \mathrm{d}^d x \Big[\frac{\Lambda}{\kappa} h - \frac{\Lambda}{4}(h^2 - 2h_{\mu\nu}h^{\mu\nu})\Big]. \tag{1-13}$$

可以看出，S_G相当于一个自由的零质量自旋-2粒子的作用量。它具有规范对称性，即在如下规范变换下保持不变：

$$\delta h_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu - \kappa(\epsilon^\lambda \partial_\lambda h_{\mu\nu} + h_{\lambda\nu}\partial_\mu \epsilon^\lambda + h_{\mu\lambda}\partial_\nu \epsilon^\lambda). \tag{1-14}$$

下一步，我们用标准的路径积分手续对其作量子化。为此，首先需要选定规范固定条件。我们选取Lorentz协变的谐和规范：

$$F_\lambda = \partial^\mu h_{\mu\lambda} - \tfrac{1}{2}\partial_\lambda h. \tag{1-15}$$

这相当于在Lagrangian中加入如下的规范固定项：

$$\mathcal{L}_{\mathrm{GF}} = \tfrac{1}{2\alpha}(\partial^\mu h_{\mu\lambda} - \tfrac{1}{2}\partial_\lambda h)^2. \tag{1-16}$$

在此规范下引力子的传播子将获得其最简单的形式，为：

$$G_{\mu\nu\alpha\beta} = \frac{i}{k^2}\Big[(2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}) - (1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)}}{k^2}\Big]. \tag{1-17}$$

为书写简洁起见，此处和今后，我们省去分母上的$+i\epsilon$项。

当考虑进宇宙学项S_Λ后，该传播子需要修改。其具体形式将在下文给出。

对于物质场作用量$S_{\rm matter}$，我们的处理手段与以上对纯引力的方法相仿，亦即对度规做微扰展开。其详细步骤亦将在下文演示。

我们既然已经获得了处理引力的规范场论框架，下一步就可以将其应用于对量子引力性质的研究。这当然是一个内容丰富的领域，因此，我们需要从一个具体问题入手。本文的主题，是在规范理论中研究量子引力对相互作用强度随能量跑动的修正。下面，让我们简单回顾这一问题的背景。

1.3 本文背景与研究之动机

非Abel规范理论的渐进自由是场论中的著名结果。此结论最初由't Hooft、Politzer[5]、 D. Gross和F. Wilczek[6]独立发现。从技术上讲，在相互作用理论中，重整化的耦合常数g之大小并不固定，而是随能量尺度μ变化。人们通常使用β函数描写这种变化，即：

$$\beta(g)=\mu\frac{{\rm d}g(\mu)}{{\rm d}\mu}. \tag{1-18}$$

例如，对于规范群为$SU(N)$的非Abel规范理论，其β函数在一圈图近似下的微扰计算结果为：

$$\beta(g)=-\frac{11Ng^3}{48\pi^2}. \tag{1-19}$$

此式右边表达式的符号起到决定性地重要作用。对于非Abel规范理论，此处符号为负，这意味着随着能量尺度的增大，耦合常数将趋于零。此即（紫外）渐进自由的含义。作为对比，量子电动力学(QED)的β函数的一圈图微扰结果为：

$$\beta(e)=\frac{e^3}{12\pi^2} \tag{1-20}$$

右端的正号意味着该理论中没有渐进自由，相似的情况出现在phi-4模型的β函数中：

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}. \tag{1-21}$$

即，phi-4模型亦无渐进自由。

由于引力与一切携带能量的物质发生相互作用。因此当我们考虑引力效应时，一切场论都将无可避免地接受修正。因此，一个十分有趣的问题是，引力

效应将如何修改其它相互作用，特别是其它三种自然界的基本相互作用？更具体地说，β函数将如何改变？

2007年，Robinson和Wilczek指出[7]，当考虑进引力效应后，QED的β函数将显示出渐进自由。它们发现，引力修正的β函数为：

$$\beta(e) = -\frac{3\kappa^2 e\mu^2}{16\pi^2}, \tag{1-22}$$

其中μ是一个特征能量尺度。

然而，这一结果很快被人们发现是规范依赖的[8]，因此在物理上是不可接受的。

为解决该问题，Toms等人使用了Vilkovisky-DeWitt有效作用量方法对该问题在标量理论和QED中重新进行研究，以期取得规范不变的结果[9][10][11]。在这些文章中，他们使用了维数正规化的方法处理发散。然而，由于维数正规化只能取出对数发散的项。而引力对规范场自能修正的领头项是二次发散，高于对数发散。因此Toms等人并未得到完整的结果。

为了得到正确且完整的领头阶量子引力修正，本文采用截断正规化的方式处理二次发散①，并将其应用于包括QED和Yang-Mills理论在内的所有规范理论，以探索量子引力对这些理论的修正。我们发现，所有规范理论在经过量子引力修正后都表现出渐进自由，并且，在一圈图近似下，最高阶引力修正的形式与规范群的具体结构无关。

① 由于Vilkovisky-DeWitt方法自身可以保证结果的规范不变性，因此在这里使用截断正规化是合理的。

第 2 章　Vilkovisky-DeWitt方法：几何有效作用量

本文的主要任务是研究引力效应对各种相互作用耦合常数随能量尺度跑动行为的修正。我们将会看到，在此过程中，几何有效作用量方法起到了关键作用。历史上，这套方法及其相关理论首先由Vilkovisky和DeWitt在二十世纪八十年代系统提出，以解决传统有效作用量方法的规范依赖问题[12] [13] [14]。因此该方法在文献中亦被广泛称为Vilkovisky-DeWitt方法。“几何有效作用量(geometrical effective action)”一词，见于Toms[15]。我们采用此术语，以突显该方法的核心思想：在场空间中置入几何结构。

本章的主题是较系统地介绍几何有效作用量方法。我们将仔细解释建立此套方法关键步骤的动机，以使物理图像尽可能地清楚。

2.1　传统有效作用量方法

前已提及，几何有效作用量方法，或Vilkovisky-DeWitt方法，是传统有效作用量方法的自然发展。因此，我们在本节中首先回顾传统有效作用量方法的基本内容。

传统有效作用量方法目前已成为场论的标准内容，关于其详细介绍，可见Peskin[16]与Weinberg[17]。

2.1.1　动机与定义

经典力学中，作用量$S[\phi]$作为场量ϕ_i的泛函，居于整个理论的核心。场的物理位形$(\phi_{\text{phys}})_i$ 可由作用量$S[\phi]$取极值之条件而获得，即：

$$\frac{\delta S[\phi]}{\delta \phi_i}\bigg|_{\phi_i=(\phi_{\text{phys}})_i} = 0. \tag{2-1}$$

正是作用量$S[\phi]$的这一性质使它在经典理论显得极为重要。

然而，在量子理论中，由于量子涨落，作用量的这个良好性质消失了。因为一般说来，量子修正将使场的物理位形发生改变，从而与经典方程所决定的物理位形相异。尽管如此，我们仍然希望寻找作用量泛函的替代品，或者说它的量子版本，使得作用量在经典理论中的良好性质仍然能被量子理论继承下来。事实上，这个替代者是存在的，它称为有效作用量。

现在，我们为有效作用量赋予精确定义。为此，从路径积分$Z[J]$出发，

$$Z[J] = \langle 0|0\rangle_J = e^{\mathrm{i}W[J]} = \int \mathcal{D}\phi \, \exp\left[\mathrm{i}S[\phi] + \mathrm{i}J_i\phi_i\right]. \tag{2-2}$$

注意$Z[J]$是外源J_i的泛函. 我们可以将场变量ϕ_i在外源J_i下之真空期望值(VEV)通过该路径积分表达出来，即：

$$\langle\phi_i\rangle \equiv \frac{\langle 0|\phi_i|0\rangle_J}{\langle 0|0\rangle_J} = -\frac{\mathrm{i}}{Z[J]}\frac{\delta Z[J]}{\delta J_i} = \frac{\delta W[J]}{\delta J_i}. \tag{2-3}$$

反之，若将真空期望值$\langle\phi_i\rangle = \bar{\phi}_i$视为已知量，则外源$J_i$亦可被视为$\bar{\phi}_i$的泛函，$J_i$对$\phi_i$的依赖关系可通过以下关系式读出：

$$\left.\frac{\delta W[\tilde{J}]}{\delta \tilde{J}_i}\right|_{\tilde{J}_i = J_i} = \bar{\phi}_i. \tag{2-4}$$

留心此式，我们即可定义有效作用量$\Gamma[\bar{\phi}]$为连通Green函数生成泛函$W[J]$的Legendre变换：

$$\Gamma[\bar{\phi}] = W[J] - J_i\bar{\phi}_i. \tag{2-5}$$

根据Legendre变换的定义，$\Gamma[\bar{\phi}]$是真空期望值$\bar{\phi}_i$的泛函。由于$\bar{\phi}_i$在此时被视为预先给定的，我们在下文中亦称之为外场，或背景场。

$\Gamma[\bar{\phi}]$之所以被称为有效作用量，是由于它满足如下关系：

$$\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}_i} = -J_i. \tag{2-6}$$

当所有的外源$J(x)$被关闭，此关系式即与经典的作用量原理具有相同的形式。这正是我们所希望看到的性质，因此$\Gamma[\bar{\phi}]$ 的确可被当作经典作用量的量子版本。

2.1.2 有效作用量的微扰计算

实际计算需要有效作用量的微扰展开式。现推导之[18]。方便起见，定义场量ϕ_{cl_i}为外源J_i存在时经典运动方程之解，并称之为经典场。即有：

$$\left.\frac{\delta}{\delta\phi_i}S[\phi]\right|_{\phi_i = \phi_{\mathrm{cl}_i}} = -J_i. \tag{2-7}$$

从而，我们可在ϕ_{cl_i}附近对$S[\phi_i]$做Taylor展开：

$$S[\phi_{\mathrm{cl}} + \phi] = S[\phi_{\mathrm{cl}}] + \phi_i \left.\frac{\delta}{\delta\phi_i}S[\phi]\right|_{\phi=\phi_{\mathrm{cl}}}$$

$$+\tfrac{1}{2}\phi_i\phi_j\frac{\delta^2}{\delta\phi_i\delta\phi_j}S[\phi]\Big|_{\phi=\phi_{\mathrm{cl}}}+I[\phi_{\mathrm{cl}};\phi], \tag{2-8}$$

其中，我们用$I[\phi_{\mathrm{cl}},\phi]$表示$\phi$的高阶项. 现在，按下式定义传播子$D$:

$$\mathrm{i}D_{ij}^{-1}=\frac{\delta^2}{\delta\phi_i\delta\phi_j}S[\phi]\Big|_{\phi=\phi_{\mathrm{cl}}}, \tag{2-9}$$

同时注意到(2-7)式，即可得到：

$$S[\phi_{\mathrm{cl}}+\phi]=S[\phi_{\mathrm{cl}}]-J_i\phi_i+\tfrac{1}{2}\phi_i(iD_{ij}^{-1})\phi_j+I[\phi_{\mathrm{cl}},\phi]. \tag{2-10}$$

从而，

$$\begin{aligned}Z[\phi]&=\int\mathcal{D}\phi\exp\left[\mathrm{i}S[\phi_{\mathrm{cl}}+\phi]+\mathrm{i}J_i(\phi_{\mathrm{cl}i}+\phi_i)\right]\\&=\exp\left[\mathrm{i}S[\phi_{\mathrm{cl}}]+\mathrm{i}J_i\phi_{\mathrm{cl}i}\right]\int\mathcal{D}\phi\exp\left(\tfrac{\mathrm{i}}{2}\phi_i(iD_{ij}^{-1})\phi_j+I[\phi_{\mathrm{cl}}+\phi]\right)\\&=\exp\left(\mathrm{i}S[\phi_{\mathrm{cl}}]+\mathrm{i}J_i\phi_{\mathrm{cl}i}\right)\mathrm{Det}^{-1/2}(\mathrm{i}D_{ij}^{-1})\\&\quad\times\frac{\int\mathcal{D}\phi\exp\left[\tfrac{\mathrm{i}}{2}\phi_i(iD_{ij}^{-1})\phi_j+I[\phi_{\mathrm{cl}}+\phi]\right]}{\int\mathcal{D}\phi\exp\left[\tfrac{\mathrm{i}}{2}\phi_i(iD_{ij}^{-1})\phi_j\right]}\\&\equiv\exp\left(\mathrm{i}S[\phi_{\mathrm{cl}}]+\mathrm{i}J_i\phi_{\mathrm{cl}i}\right)\mathrm{Det}^{-1/2}(\mathrm{i}D_{ij}^{-1})Z_2[J],\end{aligned} \tag{2-11}$$

用连通Green函数之生成泛函来表达，即有：

$$W[J]=S[\phi_{\mathrm{cl}}]+J_i\phi_{\mathrm{cl}i}+\tfrac{\mathrm{i}}{2}\log\,\mathrm{Det}\,(iD_{ij}^{-1})-\mathrm{i}\log Z_2[J]. \tag{2-12}$$

于是，可通过对$W[J]$作Legendre变换以获得有效作用量$\Gamma[\bar\phi]$的微扰展开。为此，我们定义经典场$\phi_{\mathrm{cl}i}$与期望值（背景场）$\bar\phi_i$之差为ϕ_i^1，

$$\phi_{\mathrm{cl}i}=\bar\phi_i+\phi_i^1. \tag{2-13}$$

由于经典场与背景场之差完全由量子涨落所致，故而ϕ_i^1作为背景场$\bar\phi_i$的泛函，与Planck常量$\hbar$同量级. 注意到此点即不难发现，将有效作用量按ϕ_i^1展开即等效于按$\hbar$展开。具体而言，我们有：

$$\begin{aligned}\Gamma[\bar\phi]&=W[J]-J_i\bar\phi_i\\&=S[\phi_{\mathrm{cl}}]+J_i(\phi_{\mathrm{cl}i}-\bar\phi_i)+\tfrac{\mathrm{i}}{2}\log\,\mathrm{Det}\,(\mathrm{i}D_{ij}^{-1})-\mathrm{i}\log Z_2[J]\\&=S[\bar\phi+\phi^1]+J_i\phi_i^1+\tfrac{\mathrm{i}}{2}\log\,\mathrm{Det}\,(\mathrm{i}D_{ij}^{-1})-\mathrm{i}\log Z_2[J]\end{aligned}$$

$$
\begin{aligned}
&= S[\bar{\phi}] + \phi_i^1 \frac{\delta S[\phi]}{\delta \phi}\Big|_{\phi=\bar{\phi}} + \tfrac{1}{2}\phi_i^1\phi_j^1 \frac{\delta^2 S[\phi]}{\delta\phi_i\delta_j}\Big|_{\phi=\bar{\phi}} \\
&\quad + \tfrac{\mathrm{i}}{2}\log\,\mathrm{Det}\,(\mathrm{i}D_{ij}^{-1}) - \mathrm{i}\log Z_2[J] + J_i\phi_i^1 \\
&= S[\bar{\phi}] + \tfrac{\mathrm{i}}{2}\log\,\mathrm{Det}\,(\mathrm{i}D_{ij}^{-1}) + O(\hbar^2).
\end{aligned} \tag{2-14}
$$

最后一行即按$\hbar$展开到一次项的结果。如所周知，按$\hbar$展开等价于按圈图展开。因此上式也是近似到一圈图的结果。

2.2 几何有效作用量

本节与下节之内容源自Parker & Toms[15]。

2.2.1 传统有效作用量方法之缺陷

为理解上文所述的传统有效作用量（此处冠以“传统”二字以与本节的几何有效作用量相区别）方法可能的不足之处，我们仍然从对经典作用量的分析入手。只是，我们着眼于经典作用量的另一性质：对称性。

如所周知，作用量$S[\phi]$是时空广义坐标变换下的标量，同时也是平直空间中Lorentz变换的标量。

另外，$S[\phi]$具有另一种更大的对称性——它在场空间的场量重参数化变换下亦保持不变。由此我们可以说，作用量不仅是时空中的标量，也是场空间中的标量。这一事实可被表达为：

$$
S'[\phi'] = S[\phi]. \tag{2-15}
$$

其中，$\phi' = \phi'(\phi)$是场空间中任意的重参数化变换（当然我们假设此变换性质足够良好，比如非退化）。

作用量在场空间重参数化下的不变性，仅仅反映了这样的事实，即对同一个理论用不同的变量进行描写，其效果是等价的。在经典理论中，这是一个平庸的陈述。

然而，此问题在量子理论中就变得微妙起来。这不难理解：由于量子理论通常是由一种经典理论经量子化而来，该过程用正则量子化的语言来说，即我们需要选定一组正则坐标并用其构建对易关系。因此，即使从同一种经典理论出发，选取不同的正则变量亦将导致不同的量子理论。换言之，量子化后的理论即失去了原经典理论的重参数化不变性。

的确，依上节方法所得的传统有效作用量$\Gamma[\phi]$并无场量重参数化变换的不变性，换言之，它并不是场空间中的标量。我们可从下式理解这一事实：

$$
\begin{aligned}
e^{\mathrm{i}\Gamma[\bar{\phi}]} &= \exp\left(\mathrm{i}W[J] - \mathrm{i}J_i\bar{\phi}_i\right) \\
&= \int \mathcal{D}\phi \, \exp\left[\mathrm{i}S[\phi] - \mathrm{i}\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}_i}(\phi_i - \bar{\phi}_i)\right]. \qquad (2\text{-}16)
\end{aligned}
$$

在最后一行的表达式中，出现了两个场量的差。或者说，场空间中两点的坐标差。正是此坐标差导致有效作用量在场量重参数化下不再表现为标量。因为在一般情形下，坐标差不是矢量，故而在坐标变换下并不协变（除非空间本身是平坦的）。

这本身并不成问题。一种量子理论所携带的信息多于其相应的经典理论，因此不同的量子理论可能具有相同的经典极限，这是我们熟知的事实。

然而，在具有规范对称性的理论中，规范变换本身亦可被视作场量的重参数化。因此，量子理论中重参数化不变性的丢失也许会导致（虽然并非必定导致）规范不变性的丢失。其结果是，有效作用量$\Gamma[\bar{\phi}]$将依赖于规范选取。

这似乎也不是问题。因为有效作用量本身并非物理可观测量，因此它可以依赖于规范自由度。我们所要求的仅仅是，物理结果，例如S矩阵元，必须规范不变。的确，可以证明，即使传统有效作用量是规范依赖的，但由它导出的S矩阵元却是规范无关的[19]。

不过，除了S矩阵元，我们还对另一些物理量感兴趣，比如β函数。由于β函数反映了物理的耦合参数随能量尺度的跑动行为，因此它应当是规范无关的。但是，由规范依赖的传统有效作用量所求出的β函数是否一定是规范不变的呢？

不幸的是，β函数的规范不变性在传统有效作用量方法中的确无法保证。我们在下文将会举例演示，传统有效作用量方法的确会导致规范依赖的β函数。

为解决此问题，我们不妨对有效作用量施加更强的限制：即要求它与经典作用量一样，也是场空间中重参数化变换下的标量。在下一小节中，我们将定义这样的有效作用量。

2.2.2 几何有效作用量之定义与圈图展开

据以上讨论，我们已经清楚，传统有效作用量$\Gamma[\bar{\phi}]$的非标量性来自场空间中坐标差$(\phi^i - \bar{\phi}^i)$的非矢量性，如图2.1所示。① 正如上文所及，流形上某点的

① 从现在起，我们将在场空间中引入度规结构，因此有必要区分场量的上下指标。

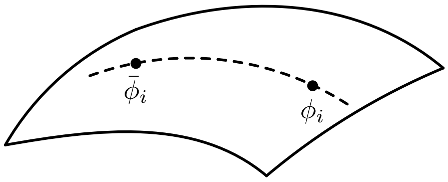


图 2.1　场空间两点的之坐标差示意图

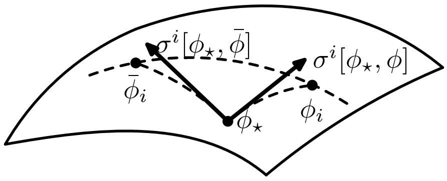


图 2.2　异地坐标差与同点处的矢量差示意图

切矢量实为其切空间中的矢量。若流形自身是平坦的线性空间，则可将它与其上一点的切空间等同起来。在此情形下，流形上两点的坐标差自然就是一个切矢量。然而，一旦流形本身是弯曲的，我们就无法作这种等同，此时坐标差不再是矢量，故而不再协变。于是可以想到，解决该问题的一种方法是，将场空间的坐标差代之以一个协变的量。

设想我们已在场空间中定义了度量G_{ij}，这样便可谈论诸如联络、长度、测地线等等概念。特别是，坐标差$(\phi^i - \bar{\phi}^i)$ 可被替换为一个协变量。详言之，首先选取场空间中另一点${\phi_\star}^i$，并设连接两点${\phi_\star}^i$与ϕ^i的测地线之长度为 $L[\phi_\star, \phi]$，继而，可依下式定义$\sigma[\phi_\star, \phi]$：

$$\sigma[\phi_\star, \phi] = \frac{1}{2} L^2[\phi_\star, \phi]. \tag{2-17}$$

于是，我们可以将$(\phi_i - \phi_{\star i})$替换为$-\sigma^i[\phi_\star, \phi]$，后者的定义是：

$$\sigma^i[\phi_\star, \phi] = G^{ij} \sigma_{,j}[\phi_\star, \phi], \tag{2-18}$$

其中，泛函导数作用于$\sigma[\phi_\star, \phi]$的第一个变量上. ① 同理，可以定义$\sigma^i[\phi_\star, \phi]$.不难看出，$\sigma^i$在其第一个变量处表现为矢量、在第二个变量处表现为标量。于是，若将坐标差$(\phi^i - \bar{\phi}^i)$代之以$\sigma^i[\phi_\star, \bar{\phi}] - \sigma^i[\phi_\star, \phi]$，则由于后者为同一点（即${\phi_\star}^i$）处两个矢量之差，故而仍是该点的矢量，具有显式协变性，如图2.2所示。 此时，若J_i在$\phi_{\star i}$点处之坐标变换下表现为矢量、同时独立于ϕ_i点（即在ϕ_i点的坐标变换下保持不变），则组合$J_i(\sigma^i[\phi_\star, \bar{\phi}] - \sigma^i[\phi_\star, \phi])$自然是重参数化变换协变的。

① 在本章中，从现在起，我们将用逗号表示关于场量的泛函偏导。

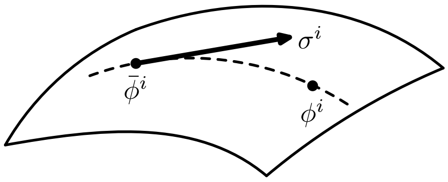


图 2.3　令$\phi^i_\star = \bar{\phi}^i$的场空间示意图

这启发我们重新定义（几何化的）路径积分泛函Z_G，为：

$$Z_\mathrm{G}[J,\phi_\star] = \exp\left(\mathrm{i}W_\mathrm{G}[J,\phi_\star]\right) = \int \mathcal{D}\phi\,\sqrt{|G[\phi]|}\exp\left[\mathrm{i}S[\phi] - iJ_i\sigma^i[\phi_\star,\phi]\right]. \tag{2-19}$$

注意，这里Z_G和W_G不仅是外源J_i的泛函，同时依赖于${\phi_\star}^i$.

由此，几何有效作用量Γ_G可被定义成$W_\mathrm{G}[J]$的Legendre变换：

$$\Gamma_\mathrm{G}[J,\phi_\star] = W_\mathrm{G}[J,\phi_\star] + J_i\sigma^i[\phi_\star,\bar{\phi}]. \tag{2-20}$$

于是，相较于此前所得的非协变有效作用量表达式：

$$e^{\mathrm{i}\Gamma[\bar{\phi}]} = \int \mathcal{D}\phi\,\exp\left[\mathrm{i}S[\phi] - \mathrm{i}\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}^i}(\phi^i - \bar{\phi}^i)\right]. \tag{2-21}$$

此时有

$$e^{\mathrm{i}\Gamma_\mathrm{G}[\bar{\phi},\phi_\star]} = \int \mathcal{D}\phi\,\sqrt{|G[\phi]|}\,\exp\left[\mathrm{i}S[\phi] - \mathrm{i}\frac{\delta\Gamma[\bar{\phi},\phi_\star]_\mathrm{G}}{\delta\sigma^i[\phi_\star,\bar{\phi}]}(\sigma^i[\phi_\star,\phi] - \sigma^i[\phi_\star,\bar{\phi}])\right]. \tag{2-22}$$

显见，按此方式定义的几何有效作用量Γ_G具有场空间中的显式协变性。

值得指出，这里的Γ_G依赖于场空间中一任意点$\phi_\star$。然而，可以证明，由此作用量所计算的物理结果与该点的选取无关[15]。因而，我们的选取以方便计算为标准。

同样为便于计算，我们需要该几何有效作用量的圈图展开。其推导与前文对传统有效作用量的圈图展开类似。根据DeWitt，选取$\phi_\star = \bar{\phi}$可有效简化计算，如图2.3所示。 在此选取下Γ_G的圈图展开至一圈图的结果为：

$$\Gamma_\mathrm{G}[\bar{\phi}] = S[\bar{\phi}] + \tfrac{\mathrm{i}}{2}\log\,\mathrm{Det}\left[S_{,ij} - \Gamma^k_{ij}S_{,k}\right], \tag{2-23}$$

其中Γ^k_{ij}即为由场空间度规G_{ij}所确定的联络。在非规范理论中，它就是Christoffel联络：

$$\Gamma^k_{ij} = \tfrac{1}{2}G^{kl}(G_{li,j} + G_{lj,i} - G_{ij,l}). \tag{2-24}$$

但是在规范理论中，Γ^k_{ij}的形式更为复杂。这种复杂性与规范理论的特殊性质有关。我们在下节讨论这一问题。

2.3 规范理论中的几何有效作用量

具有规范对称性的理论拥有许多独特的性质，因此需要单独处理。规范理论之所以特殊，主要是由于其场空间的参数化涉及两类不同的变量：物理的变量和规范的变量。其中，前者标记物理自由度，而后者标记的是多余的、因而是非物理的规范自由度。在处理规范自由度时，我们需要特别小心。在本节中，我们概述将Vilkovisky-DeWitt方法应用于规范理论时的诸多要点。

我们只考虑连续的规范对称性。在这种情形下，存在一组作用于场变量ϕ_i的连续规范变换，其无穷小形式可被写成：

$$\delta\phi^i = \phi^i_\epsilon - \phi^i = K^i_\alpha[\phi]\delta\epsilon^\alpha. \tag{2-25}$$

其中，我们用无穷小量ϵ^α对规范变换进行参数化，而K^i_α即为规范变换的生成元。我们用希腊字母 $\alpha, \beta, \cdots$ 标记规范自由度。

于是，经典作用量$S[\phi]$是规范不变量这一事实即可表述为：

$$S_{,i}[\phi]K^i_\alpha[\phi] = 0. \tag{2-26}$$

2.3.1 场空间的轨道分解(Orbit Decomposition)

如果存在规范变换，将场空间中的点ϕ^i_1变为点ϕ^i_2，则称这两点规范等价，并记之为$\phi^i_1 \sim \phi^i_2$。由此可定义场空间中的规范等价类，即：

$$[\phi^i] = \{\varphi^i;\ \varphi^i \sim \phi^i\}. \tag{2-27}$$

在物理文献中，规范等价类通常被称为“轨道(orbit)”。如果我们记整个场空间为$\mathscr{F}$，则有商空间 $\mathscr{F}/\sim$。我们称之为轨道空间(orbit space)。该商空间中的点即一条轨道，它可被看做对物理位形的唯一标记。

同时，每条轨道自身是由规范自由度构成的空间，记此空间为$\mathscr{G}$。于是我们可以说，$(\mathscr{F}/\sim)\times\mathscr{G}$（至少在局部）同构于$\mathscr{F}$。换言之，我们已经将整个场空间局部地分解成了物理的轨道空间$\mathscr{F}/\sim$与规范空间$\mathscr{G}$的直和。

一旦场空间获得此分解，则其上定义的各种量亦可作相应分解。在实际操作中，投影算子$P^i{}_j$起到了关键的作用。按定义，这组算子满足如下性质：

$$P^i{}_j K^j_\alpha = 0; \tag{2-28a}$$

$$K^i_\alpha G_{ij} P^j_k = 0; \tag{2-28b}$$

$$P^i{}_j P^j{}_k = P^i{}_k. \tag{2-28c}$$

若定义规范空间$\mathscr{G}$中的诱导度规$\gamma_{\alpha\beta}$为：

$$\gamma_{\alpha\beta} = K^i_\alpha G_{ij} K^j_\beta, \tag{2-29}$$

同时定义其逆$\gamma^{\alpha\beta}$为：

$$\gamma^{\alpha\beta}\gamma_{\beta\gamma} = \delta^\alpha_\gamma, \tag{2-30}$$

则可验证，下式给出的投影算子$P^i{}_j$满足上列各条性质。

$$P^i{}_j = \delta^i_j - K^i_\alpha \gamma^{\alpha\beta} K^k_\beta g_{kj}. \tag{2-31}$$

首先考虑场空间中任易无穷小位移的分解。显然，此位移可被分解成规范空间中的分量与正交于规范空间的分量。据定义，前者恰为一无穷小规范变换，故而可以写作：

$$\delta_\parallel \phi^i = K^i_\alpha[\phi]\delta\epsilon^\alpha. \tag{2-32}$$

而后者，作为轨道空间中的位移，可通过投影算子$P^i{}_j$方便地表示为：

$$\delta_\perp \phi^i = P^i{}_j \delta\phi^j. \tag{2-33}$$

从而，场空间的无穷小位移$\delta\phi^i$获得了如下形式的分解：

$$\delta\phi^i = \delta_\parallel\phi^i + \delta_\perp\phi^i = P^i{}_j\delta\phi^i + K^i_\alpha[\phi]\delta\epsilon^\alpha. \tag{2-34}$$

类似地，场空间的度规G_{ij}也可作这样的分解。我们已在上文中给出的该度规在规范空间所诱导度规 $\gamma_{\alpha\beta}$的定义，而轨道空间中的诱导度规$G^\perp_{ij}$也可相应地定义为：

$$G^\perp_{ij} = P^k{}_i P^l{}_j G_{kl}. \tag{2-35}$$

由此，场空间中的线元2-形式亦可被分解为：

$$\mathrm{d}s^2 = G^{\perp}_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j + \gamma_{\alpha\beta}\mathrm{d}\epsilon^\alpha\mathrm{d}\epsilon^\beta. \tag{2-36}$$

2.3.2 规范条件

到目前为止，我们一直在处理整个场空间。但是在很多情况下（比如路径积分），我们需要从每一条轨道中选取一个代表元。而作此选取的规则，即称为规范条件。

一般地，规范条件$F^\alpha[\phi]$是定义在规范空间中的泛函。我们需要它满足这样的性质，即如下方程：

$$F^\alpha[\phi] = 0 \tag{2-37}$$

在每一条轨道中有且仅有一个解。此性质在局部等价于，F^α是规范空间的正则参数化。更技术化一些，如果我们定义

$$Q^\alpha_\beta = F^\alpha_i K^i_\beta, \tag{2-38}$$

则以上条件亦等价于$\det Q^\alpha_\beta \neq 0$. 满足此性质的泛函$F^\alpha[\phi]$在局部上都是可接受的规范条件。只是在全局上，该条件尚不能保证方程之解的唯一性。事实上，在Yang-Mills理论中，的确存在这种局部唯一而全局不唯一的规范条件，文献中称之为Gribov任意性(Gribov ambiguity)。不过我们对它并不关心，因为微扰论只与局部性质有关，所以在此处我们只关心局部性质。

可见，如果我们要在场空间中选定一个坐标系（即一种场的参数化），则规范条件F^α恰好可以用来作为规范空间的坐标。与之相应，我们也可以在轨道空间中选定一组坐标ξ^A。这里我们使用大写拉丁字母标记轨道空间的自由度。

2.3.3 规范理论的场空间联络

现在，我们研究当场空间中存在规范自由度时，其上的联络会发生何种变化。

经典作用量和几何有效作用量都是规范无关量。作为定义在场空间上的泛函，规范独立即意味着它只与轨道空间的坐标有关，而与规范空间的坐标无关。

另一方面，在规范理论中，对场空间联络的选取较非规范理论有更大的余地。这是因为，我们此时并不要求整个场空间的度规G_{ij}满足平移条件$\mathrm{D}_i G_{jk} = 0$，而只需轨道空间度规$G^{\perp}_{ij}$满足之即可。设联络为$\tilde{\Gamma}^k_{ij}$，则轨道空间之度规的平移条件为：

$$\tilde{\mathrm{D}}_i G^{\perp}_{jk} \equiv G^{\perp}_{jk,i} - \tilde{\Gamma}^l_{ik} g^{\perp}_{jl} - \tilde{\Gamma}^l_{ij} G^{\perp}_{kl} = 0. \tag{2-39}$$

经过简单的代数计算可得：

$$\tilde{\Gamma}^l_{ij} G^{\perp}_{lk} = \frac{1}{2}(G^{\perp}_{jk,i} + G^{\perp}_{ik,j} - G^{\perp}_{ij,k}). \tag{2-40}$$

根据Christoffel联络的标准推导，我们应该在此式两端乘以度规的逆$\tilde{\Gamma}^l_{ij}$。然而此时这种方法不成立，因为$G^{\perp}_{ij}$，作为完整度规在轨道空间的投影，在整个场空间中并不可逆。这暗示我们，在规范理论中，场空间之联络$\tilde{\Gamma}^k_{ij}$并非简单的Christoffel形式。可以证明[15]，此时的联络系数可以表示为：

$$\begin{aligned}\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} &- \gamma^{\alpha\beta} K_{\alpha i} K^k_{\beta;j} - \gamma^{\alpha\beta} K_{\alpha j} K^k_{\beta;i} \\ &+ \frac{1}{2}\gamma^{\alpha\gamma}\gamma^{\beta\delta} K_{\alpha i} K_{\beta j}(K^l_{\gamma} K^k_{\delta;l} + K^k_{\delta} K^l_{\gamma,l}) + K^k_{\alpha} A^{\alpha}_{ij}.\end{aligned} \tag{2-41}$$

其中，第一项Γ^k_{ij}即为度规G_{ij}所诱导的Christoffel联络，而其后与规范变换之生成元K^i_{α}有关的项皆为规范理论所特有的项。值得指出，其中正比于K^i_{α}的项（即最后一项）将不贡献进几何有效作用量，因而系数A^{α}_{ij}的具体形式并不重要。以下，我们将无视此项，直接写出：

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + T^k_{ij}. \tag{2-42}$$

其中T^k_{ij}代表规范理论所特有的项：

$$\begin{aligned}T^k_{ij} = &- \gamma^{\alpha\beta} K_{\alpha i} K^k_{\beta;j} - \gamma^{\alpha\beta} K_{\alpha j} K^k_{\beta;i} \\ &+ \frac{1}{2}\gamma^{\alpha\gamma}\gamma^{\beta\delta} K_{\alpha i} K_{\beta j}(K^l_{\gamma} K^k_{\delta;l} + K^k_{\delta} K^l_{\gamma,l}).\end{aligned} \tag{2-43}$$

由以上结果，即可推导规范理论中几何有效作用量$\Gamma_{\mathrm{G}}[\bar{\phi}]$的微扰展开式。如所周知，对于规范理论而言，将其量子化的过程需要做规范固定。其效果是在量子化的作用量中加入了规范固定项S_{GF}与鬼项S_{ghost}。将这些内容一并考虑进来，可以导出，展开至一圈图的几何有效作用量$\Gamma_{\mathrm{G}}[\bar{\phi}]$为：

$$\Gamma_{\mathrm{G}}[\bar{\phi}] = S[\bar{\phi}] + S_{\mathrm{GF}} + S_{\mathrm{ghost}}$$

$$+\frac{\mathrm{i}}{2}\log\,\mathrm{Det}\left[(S+S_{\mathrm{GF}})_{,ij}-\tilde{\Gamma}^{k}_{ij}(S+S_{\mathrm{GF}})\right]_{\text{back.}}. \tag{2-44}$$

2.3.4 Landau-DeWitt规范

从以上讨论可见，在规范理论中，对联络$\tilde{\Gamma}^k_{ij}$的计算因T^k_{ij}的出现而变得相当复杂。但是，由于我们计算联络的目的是计算几何有效作用量，因此有可能通过选取适当的规范固定条件以化简计算。事实上，当我们采用由下式定义的Landau-DeWitt规范条件F_α时，T^k_{ij}对$\Gamma_{\mathrm{G}}[\bar{\phi}]$无贡献。

$$F_\alpha = K^i_\alpha(\phi^i-\bar{\phi}^i). \tag{2-45}$$

这不难证明。由T^k_{ij}的表达式：

$$\begin{aligned}T^k_{ij} =& -\gamma^{\alpha\beta}K_{\alpha i}K^k_{\beta;j}-\gamma^{\alpha\beta}K_{\alpha j}K^k_{\beta;i}\\ &+\frac{1}{2}\gamma^{\alpha\gamma}\gamma^{\beta\delta}K_{\alpha i}K_{\beta j}(K^l_\gamma K^k_{\delta;l}+K^k_\delta K^l_{\gamma,l}).\end{aligned} \tag{2-46}$$

在Landau-DeWitt规范下，即有

$$T^k_{ij}(\phi^j-\bar{\phi}^j) = -\gamma^{\alpha\beta}K_{\alpha i}K^k_{\beta;j}(\phi^j-\bar{\phi}^j), \tag{2-47}$$

从而，

$$T^k_{ij}(\phi^i-\bar{\phi}^i)(\phi^j-\bar{\phi}^j) = 0. \tag{2-48}$$

而T^k_{ij}恰好是以这种组合进入几何有效作用量$\Gamma_{\mathrm{G}}[\bar{\phi}]$的，因此在Landau-DeWitt规范下，我们根本无需计算形式复杂的T^k_{ij}。这在下一章的计算中十分关键。

2.4 以Coleman-Weinberg模型为例的应用

到此为止，我们对Vilkovisky-DeWitt方法进行了形式和抽象的讨论。在将这套方法应用于较为复杂的引力理论之前，我们首先用一个相对简单的例子，即Coleman-Weinberg模型来演示以上结果。

Coleman-Weinberg模型由S. Coleman与E. Weinberg提出[20]，作为展示动力学对称性自发破缺的一个例子。然而，人们很快注意到，他们的结果是规范依赖的，从而是值得怀疑的[18]。

在本节中，我们将首先使用传统的有效作用量方法推导Coleman-Weinberg模型的有效势，以强调其结果的规范依赖性。然后，我们将使

用Vilkovisky-DeWitt方法推导几何有效势，以取得规范无关的结果。我们将在两种不同的规范条件下进行计算，分别是 Lorentz规范与Landau-DeWitt规范。

这两种规范各有优缺点。在Lorentz规范下，鬼场与其它物质场无相互作用，因此可不考虑之。同时，该规范所导致的规范固定项中含有任意规范参数ξ。因此在计算末尾ξ显式的消去即成为结果之规范无关性的非平凡检验。然而，正如上节所述，在此规范下，我们必须计算联络中复杂的T^k_{ij}。

正好相反，在Landau-DeWitt规范下，我们无需计算T^k_{ij}，但是要处理鬼场，同时由于该规范条件的限制，我们必须在计算末尾取任意的规范参数$\xi \to 0$。因此无法显式说明结果的规范无关性。

由于Coleman-Weinberg模型相对简单，因此我们可以使用这两种规范分别处理之，以期得到相同的结果。

在正式应用这些方法之前，我们首先对Coleman-Weinberg模型作简要介绍。

2.4.1 Coleman-Weinberg模型

Coleman-Weinberg实为将电子换为无质量标量粒子的量子电动力学。我们在此做一点容易的推广，即允许标量粒子也带有质量m。从而，该模型的经典Lagrangian可被写作：

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\mathrm{D}_\mu\phi)^a(\mathrm{D}^\mu\phi)^a - \frac{1}{2}m^2\phi^a\phi^a - \frac{1}{4!}\lambda(\phi^a\phi^a)^2, \tag{2-49}$$

其中，场强张量$F_{\mu\nu}$为

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2-50}$$

同时，我们使用了记号D_μ代表对时空坐标的规范协变微商：

$$(\mathrm{D}_\mu\phi)^a = \partial_\mu\phi^a - eA_\mu\epsilon^{ab}\phi^b. \tag{2-51}$$

这里，标量场的指标$a = 1, 2$，而ϵ^{ab}是二阶全反对称张量，并采用约定$\epsilon^{12} = 1$。注意，此时的标量场必须是复标量，（或等价地说，二分量标量列），以使之与电磁场耦合起来。

由此，我们看到标量通过两种顶角与规范场A_μ相耦合：

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a - \frac{1}{4!}\lambda(\phi^a\phi^a)^2$$

$$+\frac{1}{2}e^2A_\mu A^\mu\phi^a\phi^a - eA_\mu(\partial^\mu\phi^a)\epsilon^{ab}\phi^b, \tag{2-52}$$

Coleman-Weinberg模型是一个$U(1)$规范理论。其规范变换为：

$$A_\mu \to A_\mu + \partial_\mu\alpha, \qquad \phi^a \to \phi^a + e\alpha\epsilon^{ab}\phi^b. \tag{2-53}$$

容易验证，该模型的作用量在此变换下的确保持不变。

在以下两小节，我们分别使用传统有效势方法与Vilkovisky-DeWitt的几何有效势方法去计算Coleman-Weinberg模型的有效势。所谓有效势，即取常值场变量（不依赖于时空坐标）的有效作用量。此时，作用量中的微商项为零，从而作用量从泛函退化为普通的函数。

2.4.2 传统方法

我们首先使用传统方法计算一圈图水平的有效势。

按照处理规范理论的标准程序，在量子化该理论时，需要对其做规范固定。我们使用如下的Lorentz规范：

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{2-54}$$

为计算有效势，我们打开标量场ϕ的外源，使其获得一个非零的常数背景$\bar{\phi}$（即与时空坐标无关），同时保持规范场A_μ 的背景值仍为零。则按照此前推导，一圈图水平的有效作用量为：

$$\Gamma[\bar{\phi}] = S[\bar{\phi}] + \frac{\mathrm{i}}{2}\log\,\mathrm{Det}\,[S_{,ij}]_{\text{back.}}, \tag{2-55}$$

其中，“back.”意为泛函偏导须取背景值。

因此，我们需计算如下对场量之二阶泛函偏导在背景场处的取值：

$$S_{,(\mu x)(\nu y)}|_{\text{back.}} = [(\partial^2 + e^2\bar{\phi}^2)g^{\mu\nu} - (1-\tfrac{1}{\xi})\partial^\mu\partial^\nu]\delta(x-y); \tag{2-56}$$

$$S_{,(ax)(by)}|_{\text{back.}} = [(-\partial^2 - m^2)\delta^{ab} - \tfrac{1}{6}\lambda\bar{\phi}^2\delta^{ab} - \tfrac{1}{3}\lambda\bar{\phi}^a\bar{\phi}^b]\delta(x-y); \tag{2-57}$$

$$S_{,(\mu x)(by)}|_{\text{back.}} = e\epsilon^{bc}\partial^\mu\bar{\phi}^c\delta(x-y). \tag{2-58}$$

为简化表达式，我们用拉丁指标a, b代表标量场ϕ^a，并用希腊指标μ, ν代表规范场A_μ，而x, y是这些场量的时空依赖。一如既往，泛函偏导用逗号表示。

以上列各式表达的泛函偏导不是对角化的形式，因此不易对其取行列式。为解决之，我们将其Fourier变换至动量空间。

在动量空间中，我们有：

$$S_{,ij} = \begin{pmatrix} \Delta_{ab} & \Delta_{a\nu} \\ \Delta_{\mu b} & \Delta_{\mu\nu} \end{pmatrix}, \tag{2-59}$$

其中：

$$\Delta_{ab} =(k^2 - m^2 - \tfrac{1}{6}\lambda\bar{\phi}^2)\delta^{ab} - \tfrac{1}{3}\lambda\bar{\phi}^a\bar{\phi}^b; \tag{2-60a}$$

$$\Delta_{\mu\nu} =(- k^2 + e^2\bar{\phi}^2)g^{\mu\nu} + (1 - \tfrac{1}{\xi})k^\mu k^\nu; \tag{2-60b}$$

$$\Delta_{\mu b} = - \Delta_{b\mu} = -\mathrm{i}ek^\mu\epsilon^{bc}\bar{\phi}^c. \tag{2-60c}$$

为计算该矩阵的行列式，我们将改写为：

$$\det\begin{pmatrix} \Delta_{ab} & \Delta_{a\nu} \\ \Delta_{\mu b} & \Delta_{\mu\nu} \end{pmatrix} = \det\Delta_{ab}\det\Big[\Delta_{\mu\nu} - \frac{\Delta_{\mu b}\Delta_{a\nu}}{\Delta_{ab}}\Big], \tag{2-61}$$

同时使用以下矩阵代数的技巧：

$$\lambda^m \det(\lambda I_n - AB) = \lambda^n \det(\lambda I_m - BA). \tag{2-62}$$

这里I_n代表$n \times n$单位矩阵，而A是$n \times m$矩阵、B是$m \times n$矩阵。λ是任意数。

于是：

$$\det\Delta_{ab} = (k^2 - m^2 - \tfrac{1}{6}\lambda\bar{\phi}^2)(k^2 - m^2 - \tfrac{1}{2}\lambda\bar{\phi}^2); \tag{2-63}$$

为计算另一个行列式因子，我们注意到：

$$(\Delta_{ab})^{-1} = \frac{1}{k^2 - m^2 - \frac{1}{6}\lambda\bar{\phi}^2}\Big[\delta^{ab} + \frac{\frac{\lambda}{3}\bar{\phi}^a\bar{\phi}^b}{k^2 - m^2 - \frac{1}{2}\lambda\bar{\phi}^2}\Big] \tag{2-64}$$

再次使用上述矩阵代数的技巧，我们有：

$$\begin{aligned}
&\det\Big[\Delta_{\mu\nu} - \frac{\Delta_{\mu b}\Delta_{a\nu}}{\Delta_{ab}}\Big] \\
&=\det\Big[(-k^2 + e^2\bar{\phi}^2)\delta^{\mu\nu} + (1 - \tfrac{1}{\xi})k^\mu k^\nu - \frac{e^2\bar{\phi}^2k^\mu k^\nu}{k^2 - m^2 - \frac{1}{6}\lambda^2\bar{\phi}^2}\Big] \\
&=(k^2 - e^2\bar{\phi}^2)^3\Big[k^2 - e^2\bar{\phi}^2 - (1 - \tfrac{1}{\xi})k^2 + \frac{e^2\bar{\phi}^2k^2}{k^2 - m^2 - \frac{1}{6}\bar{\phi}^2}\Big] \\
&=\frac{(k^2 - e^2\bar{\phi}^2)^3}{k^2 - m^2 - \frac{1}{6}\bar{\phi}^2}\Big[k^4 - (m^2 + \tfrac{1}{6}\lambda\bar{\phi}^2)k^2 + \xi(m^2 + \tfrac{1}{6}\lambda\bar{\phi}^2)e^2\bar{\phi}^2\Big].
\end{aligned} \tag{2-65}$$

于是，我们可立即写出（未重整化的）有效势，为：

$$
\begin{aligned}
V(\bar{\phi}) =& \frac{1}{4!}\bar{\phi}^4 + \frac{1}{2}\int \frac{\mathrm{d}^4 k_E}{(2\pi)^4}\Big[\log\big(1+\frac{m^2+\frac{1}{2}\lambda\bar{\phi}^2}{k_E^2}\big) + 3\log\big(1+\frac{e^2\bar{\phi}^2}{k_E^2}\big) \\
&+ \log\big(1+\frac{m^2+\frac{1}{6}\lambda\bar{\phi}^2}{k_E^2} + \frac{\xi(m^2+\frac{1}{6}\lambda\bar{\phi}^2)e^2\bar{\phi}^2}{k_E^4}\big)\Big].
\end{aligned}
\tag{2-66}
$$

可见，该结果是规范依赖的。

2.4.3 几何有效势：在Lorentz规范下的计算

本节，我们选取Lorentz规范计算Vilkovisky-DeWitt几何有效势。与此前相同，Lorentz规范固定项由下式给出：

$$
\mathcal{L}_{\mathrm{GF}} = -\tfrac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{2-67}
$$

在一圈图水平，几何有效作用量为：

$$
\Gamma[\bar{\phi}] = S[\bar{\phi}] + S_{GF}[\bar{\phi}] + \tfrac{\mathrm{i}}{2}\log\,\mathrm{Det}\,[S_{,ij} - \Gamma^k_{ij}S_{,k}]_{\mathrm{back.}}. \tag{2-68}
$$

简言之，与传统计算的区别即将泛函偏导换为泛函协变导数。

因此，我们需要计算场空间的联络。根据此前讨论，略去与有效作用量无关的正比于生成元的部分，它接受来自两部分贡献：

$$
\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + T^k_{ij}. \tag{2-69}
$$

其中，Γ^k_{ij}即为由场空间度规所诱导的Christoffel符号，而T^k_{ij}由下式给出：

$$
\begin{aligned}
T^k_{ij} =& -\gamma^{\alpha\beta}K_{\alpha i}K^k_{\beta;j} - \gamma^{\alpha\beta}K_{\alpha j}K^k_{\beta;i} \\
&+ \tfrac{1}{2}\gamma^{\alpha\gamma}\gamma^{\beta\delta}K_{\alpha i}K_{\beta j}(K^l_\gamma K^k_{\delta;l} + K^k_\delta K^l_{\gamma;l})
\end{aligned}
\tag{2-70}
$$

值得强调，其中的分号表示用Christoffel符号所计算的泛函协变微商。由于此时Christoffel符号所有分量皆为零，因此这相当于泛函偏导。

现在我们逐步计算这些项。

首先，对于Coleman-Weinberg模型，场空间的度规G_{ij}可取这样的形式，其非零分量为：

$$
G_{\phi^a(x)\phi^b(y)} = \delta_{ab}\delta(x-y); G_{A_\mu(x)A_\nu(y)} = \eta^{\mu\nu}\delta(x-y). \tag{2-71}
$$

这组度规为场空间赋予了平坦的几何。这体现在尤其诱导的Christoffel联络$\Gamma^k_{ij}=0$。因此，我们只需计算由于规范理论自身的特殊性而引入的T^k_{ij}项。

为此，从该模型的规范变换中读出其生成元K，为：

$$\begin{aligned}K_u^{\phi^a(x)} &= e\epsilon^{ab}\phi^b(x)\delta(x-y),\\ K_u^{A_\mu(x)} &= \partial_\mu\delta(x-u).\end{aligned} \tag{2-72}$$

而$\gamma_{\alpha\beta}$作为规范空间中的诱导度规，其定义为：

$$\gamma_{\alpha\beta}=G_{ij}K^i_\alpha K^j_\beta \tag{2-73}$$

因此，在我们目前大的情形，规范指标α只有唯一的取值（注意到$U(1)$是一维的Lie群），因此有：

$$\gamma=(-\partial^2+e^2\bar{\phi}^2)\delta(x-y). \tag{2-74}$$

将这些结果代入$\tilde{\Gamma}^k_{ij}=T^k_{ij}$的表达式，即可得到联络的非零分量，为：

$$\begin{aligned}\Gamma^{\phi^c(z)}_{\phi^a(x)\phi^b(y)} &= \frac{e^2(\epsilon^{bc}\epsilon^{ad}\bar{\phi}^d+\epsilon^{ac}\epsilon^{bd}\bar{\phi}^d)}{-\partial^2+e^2\bar{\phi}^2}\delta(z-x)\delta(z-y)\\ &\quad-\frac{e^4\phi^c\epsilon^{ae}\phi^e\epsilon^{bd}\phi^d}{(-\partial^2+e^2\bar{\phi}^2)^2}\delta(z-x)\delta(z-y);\end{aligned} \tag{2-75}$$

$$\Gamma^{\phi^c(z)}_{A_\mu(x)A_\nu(y)} = -\frac{e^2\bar{\phi}^c(z)}{(-\partial^2+e^2\bar{\phi}^2)^2}\partial^\mu\delta(x-z)\partial^\nu\delta(y-z); \tag{2-76}$$

$$\begin{aligned}\Gamma^{\phi^c(z)}_{A_\mu(x)\phi^b(y)} &= \frac{e\epsilon^{bc}}{-\partial^2+e^2\bar{\phi}^2}\partial^\mu\delta(x-z)\delta(y-z)\\ &\quad-\frac{e^3\bar{\phi}^c\epsilon^{bd}\bar{\phi}^d}{(-\partial^2+e^2\bar{\phi}^2)^2}\partial^\mu\delta(x-z)\delta(y-z).\end{aligned} \tag{2-77}$$

根据此前的经验，我们现在进入动量空间。注意到：

$$\nabla_i\nabla_j S=\begin{pmatrix}\tilde{\Delta}_{ab} & \tilde{\Delta}_{a\nu}\\ \tilde{\Delta}_{\mu b} & \tilde{\Delta}_{\mu\nu}\end{pmatrix}, \tag{2-78}$$

从而，

$$\begin{aligned}\tilde{\Delta}_{ab} &= \Delta_{ab}-\tilde{\Gamma}^c_{ab}S_{,c}\big|_{back}\\ &=(k^2-m^2-\tfrac{1}{6}\lambda\bar{\phi}^2)\delta^{ab}-\tfrac{1}{3}\lambda\bar{\phi}^a\bar{\phi}^b\\ &\qquad+\Big[\frac{2e^2}{-\partial^2+e^2\bar{\phi}^2}-\frac{e^4\bar{\phi}^2}{(-\partial^2+e^2\bar{\phi}^2)^2}\Big](m^2+\tfrac{1}{6}\lambda\bar{\phi}^2)(\epsilon^{ac}\bar{\phi}^c\epsilon^{bd}\bar{\phi}^d);\end{aligned} \tag{2-79}$$

$$\tilde{\Delta}_{\mu\nu} = (-k^2 + e^2\bar{\phi}^2)g^{\mu\nu} + (1 - \tfrac{1}{\xi})k^\mu k^\nu - \frac{e^2\bar{\phi}^2(m^2 + \frac{1}{6}\lambda\bar{\phi}^2)}{(k^2 + e^2\bar{\phi}^2)^2}k^\mu k^\nu; \tag{2-80}$$

$$\tilde{\Delta}_{\mu b} = -\tilde{\Delta}_{b\mu} = -ie\epsilon^{bd}\bar{\phi}^d k^\mu\Big[1 - \frac{m^2 + \frac{1}{6}\lambda\bar{\phi}^2}{k^2 + e^2\bar{\phi}^2} + \frac{e^2\bar{\phi}^2(m^2 + \frac{1}{6}\lambda\bar{\phi}^2)}{(k^2 + e^2\bar{\phi}^2)^2}\Big]. \tag{2-81}$$

从现在开始，所有的计算都与上节中的有关步骤平行，只是表达式显得复杂一些。我们需要计算如下行列式：

$$\det\begin{pmatrix} \tilde{\Delta}_{ab} & \tilde{\Delta}_{a\nu} \\ \tilde{\Delta}_{\mu b} & \tilde{\Delta}_{\mu\nu} \end{pmatrix} = \det\tilde{\Delta}_{ab}\det\Big(\tilde{\Delta}_{\mu\nu} - \frac{\tilde{\Delta}_{\mu b}\tilde{\Delta}_{a\nu}}{\tilde{\Delta}_{ab}}\Big), \tag{2-82}$$

首先，计算$\det\tilde{\Delta}_{ab}$：

$$\begin{aligned}\det\tilde{\Delta}_{ab} =& \det\Big[(k^2 - m^2 - \tfrac{1}{6}\bar{\phi}^2)\delta^{ab} - \tfrac{1}{3}\lambda\bar{\phi}^a\bar{\phi}^b + B\epsilon^{ac}\bar{\phi}^c\epsilon^{bd}\bar{\phi}^d\Big] \\ =&(k^2 - m^2 - \tfrac{1}{6}\lambda\bar{\phi}^2)(k^2 - m^2 - \tfrac{1}{2}\bar{\phi}^2 + B\bar{\phi}^2) - \tfrac{1}{3}\lambda B(\bar{\phi}^2)^2 \\ =&\frac{k^2(k^2 - m^2 - \frac{1}{2}\lambda\bar{\phi}^2)}{(k^2 + e^2\bar{\phi}^2)^2}[k^4 - (m^2 - 2e^2\bar{\phi}^2 + \tfrac{1}{6}\lambda^2\bar{\phi}^2)k^2 + e^4(\bar{\phi}^2)^2],\end{aligned} \tag{2-83}$$

其中，我们定义

$$B = \Big(\frac{2e^2}{k^2 + e^2\bar{\phi}^2} - \frac{e^4\bar{\phi}^2}{(k^2 + e^2\bar{\phi}^2)^2}\Big)(m^2 + \tfrac{1}{6}\lambda\bar{\phi}^2). \tag{2-84}$$

在以上计算中，我们同样使用了上一小节提到的矩阵技巧。

然后，计算$\det(\tilde{\Delta}_{\mu\nu} - \tilde{\Delta}_{\mu b}(\tilde{\Delta}_{ab})^{-1}\tilde{\Delta}_{a\nu})$。

为此，首先求出$\tilde{\Delta}_{ab}$的逆：

$$(\Delta_{ab})^{-1} = \frac{1}{k^2 - m^2 - \frac{1}{6}\lambda\bar{\phi}^2}\Big(\delta^{ab} + \frac{\lambda}{3}\frac{\bar{\phi}^a\bar{\phi}^b}{k^2 - m^2 - \frac{1}{2}\bar{\phi}^2} - \frac{B\epsilon^{ac}\bar{\phi}^c\epsilon^{bd}\bar{\phi}^d}{k^2 - m^2 - \frac{1}{6}\lambda\bar{\phi}^2 + B\bar{\phi}^2}\Big). \tag{2-85}$$

于是有：

$$\begin{aligned}&\tilde{\Delta}_{\mu\nu} - \tilde{\Delta}_{\mu b}(\tilde{\Delta}_{ab})^{-1}\tilde{\Delta}_{a\nu} \\ =&(-k^2 + e^2\bar{\phi}^2)g^{\mu\nu} + (1 - \tfrac{1}{\xi})k^\mu k^\nu - \frac{e^2\bar{\phi}^2(m^2 + \frac{1}{6}\lambda\bar{\phi}^2)}{(k^2 + e^2\bar{\phi}^2)^2}k^\mu k^\nu \\ &- e^2\epsilon^{ae}\bar{\phi}^e\epsilon^{bf}\bar{\phi}^f k^\mu k^\nu\Big[1 - \frac{m^2 + \frac{1}{6}\lambda\bar{\phi}^2}{k^2 + e^2\bar{\phi}^2} + \frac{e^2\bar{\phi}^2(m^2 + \frac{1}{6}\lambda\bar{\phi}^2)}{(k^2 + e^2\bar{\phi}^2)^2}\Big]^2 \\ &\times\frac{1}{k^2 - m^2 - \frac{1}{6}\lambda\bar{\phi}^2}\Big[\delta^{ab} + \frac{\lambda\bar{\phi}^a\bar{\phi}^b/3}{k^2 - m^2 - \frac{1}{2}\bar{\phi}^2} - \frac{B\epsilon^{ac}\bar{\phi}^c\epsilon^{bd}\bar{\phi}^d}{k^2 - m^2 - \frac{1}{6}\lambda\bar{\phi}^2 + B\bar{\phi}^2}\Big] \\ =&(-k^2 + e^2\bar{\phi}^2)g^{\mu\nu} + (1 - \tfrac{1}{\xi})k^\mu k^\nu - \frac{e^2\bar{\phi}^2(m^2 + \frac{1}{6}\lambda\bar{\phi}^2)}{(k^2 + e^2\bar{\phi}^2)^2}k^\mu k^\nu\end{aligned}$$

$$-e^2\bar{\phi}^2k^\mu k^\nu\Big[1-\frac{m^2+\frac{1}{6}\lambda\bar{\phi}^2}{k^2+e^2\bar{\phi}^2}+\frac{e^2\bar{\phi}^2(m^2+\frac{1}{6}\lambda\bar{\phi}^2)}{(k^2+e^2\bar{\phi}^2)^2}\Big]^2\frac{\bar{\phi}^2}{k^2-m^2-\frac{1}{6}\lambda\bar{\phi}^2+B\bar{\phi}^2}. \tag{2-86}$$

从而，再次使用求矩阵行列式的技巧，有：

$$\begin{aligned}
&\det\Big(\tilde{\Delta}_{\mu\nu}-\frac{\tilde{\Delta}_{\mu b}\tilde{\Delta}_{a\nu}}{\tilde{\Delta}_{ab}}\Big)\\
=&(k^2-e^2\bar{\phi}^2)^3\Big[k^2-e^2\bar{\phi}^2-(1-\tfrac{1}{\xi})k^2+\frac{e^2\bar{\phi}^2(m^2+\frac{1}{6}\lambda\bar{\phi}^2)k^2}{(k^2+e^2\bar{\phi}^2)^2}\\
&+\Big(1-\frac{m^2+\frac{1}{6}\lambda\bar{\phi}^2}{k^2+e^2\bar{\phi}^2}+\frac{e^2\bar{\phi}^2(m^2+\frac{1}{6}\lambda\bar{\phi}^2)}{(k^2+e^2\bar{\phi}^2)^2}\Big)^2\frac{e^2\bar{\phi}^2k^2}{k^2-m^2-\frac{1}{6}\lambda\bar{\phi}^2+B\bar{\phi}^2}\Big]\\
=&\tfrac{1}{\xi}(k^2-e^2\bar{\phi}^2)^3k^2.
\end{aligned} \tag{2-87}$$

从而，我们可以写下几何有效势为：

$$\begin{aligned}
V_{\mathrm{G}}(\bar{\phi})=&\frac{1}{4!}\bar{\phi}^4+\frac{1}{2}\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\Big[\log\Big(1+\frac{m^2+\frac{1}{2}\lambda\bar{\phi}^2}{k_E^2}\Big)+3\log\Big(1+\frac{e^2\bar{\phi}^2}{k_E^2}\Big)\\
&+\log\Big(1+\frac{m^2-2e^2\bar{\phi}^2+\frac{1}{6}\lambda^2\bar{\phi}^2}{k_E^2}-\frac{e^4(\bar{\phi}^2)^2}{k_E^4}\Big)-2\log\Big(1-\frac{e^2\bar{\phi}^2}{k_E^2}\Big)\Big]+\text{const.}
\end{aligned} \tag{2-88}$$

可见，所有对规范参数ξ的依赖在我们求行列式的过程中被化归为行列式前的系数。此系数对有效势的贡献为常数，而由是能零点的任意性，此常数是无关紧要的。故而，最终的几何有效势是规范无关的。

2.4.4 几何有效势：在Landau-DeWitt规范下的计算

现在，我们再取Landau-DeWitt规范重复以上计算。根据定义，Landau-DeWitt规范为：

$$F=K^{\phi^a(x)}\varphi^a(x)=\partial_\mu A^\mu+e\epsilon^{ab}\bar{\phi}^a(x)\varphi^b(x). \tag{2-89}$$

如前所述，在该规范下我们无需计算T^k_{ij}。因此该计算过程与传统方法相仿。唯一之区别在于，此时的规范固定项是：

$$\mathcal{L}_{\mathrm{GF}}=-\tfrac{1}{2\xi}(\partial_\mu A^\mu+e\epsilon^{ab}\bar{\phi}^a\varphi^b)^2. \tag{2-90}$$

并且，在计算末尾，须取$\xi\to 0$。

于是，代之以(2-91a)式，此时有：

$$\Delta_{ab} =(k^2 - m^2 - \tfrac{1}{6}\lambda\bar{\phi}^2)\delta^{ab} - \tfrac{1}{3}\lambda\bar{\phi}^a\bar{\phi}^b - \tfrac{1}{\xi}e^2\epsilon^{ac}\epsilon^{bd}\bar{\phi}^b\bar{\phi}^d; \tag{2-91a}$$

$$\Delta_{\mu\nu} =(-k^2 + e^2\bar{\phi}^2)g^{\mu\nu} + (1 - \tfrac{1}{\xi})k^\mu k^\nu; \tag{2-91b}$$

$$\Delta_{\mu b} = -\Delta_{b\mu} = -\mathrm{i}e(1 - \tfrac{1}{\xi})k^\mu\epsilon^{bc}\bar{\phi}^c. \tag{2-91c}$$

重复求行列式的过程，

$$\det\begin{pmatrix}\Delta_{ab} & \Delta_{a\nu}\\ \Delta_{\mu b} & \Delta_{\mu\nu}\end{pmatrix} = \det\Delta_{ab}\det\left[\Delta_{\mu\nu} - \frac{\Delta_{\mu b}\Delta_{a\nu}}{\Delta_{ab}}\right],$$

可得，

$$\det\Delta_{ab} = \tfrac{1}{2\xi}(k^2 - m^2 - \tfrac{1}{2}\lambda\bar{\phi}^2)[\xi(k^2 - m^2 - \tfrac{1}{6}\lambda\bar{\phi}^2) - e^2\bar{\phi}^2]; \tag{2-92}$$

以及，

$$\begin{aligned}&\det\left[\Delta_{\mu\nu} - \frac{\Delta_{\mu b}\Delta_{a\nu}}{\Delta_{ab}}\right]\\ &= -(e^2\bar{\phi}^2)^{-1}(k^2 - e^2\bar{\phi}^2)^3[k^4 - (m^2 + \tfrac{1}{6}\lambda\bar{\phi}^2 - 2e^2\bar{\phi}^2)k^2 + e^4(\bar{\phi}^2)^2].\end{aligned} \tag{2-93}$$

然而不要忘记，我们此时必须考虑鬼项$\mathcal{L}_{\mathrm{ghost}}$。它由下式给出：

$$\mathcal{L}_{\mathrm{ghost}} = \bar{c}(\partial^2 + e^2\bar{\phi}^2)c. \tag{2-94}$$

于是在行列式中尚需加入如下部分：

$$\det\Delta_{cc} = -(k^2 - e^2\bar{\phi}^2); \tag{2-95}$$

综上，可得此时的有效势为：

$$\begin{aligned}V_{\mathrm{G}}(\bar{\phi}) =&\frac{1}{4!}\bar{\phi}^4 + \frac{1}{2}\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\Big[\log\Big(1 + \frac{m^2 + \frac{1}{2}\lambda\bar{\phi}^2}{k_E^2}\Big) + 3\log\Big(1 + \frac{e^2\bar{\phi}^2}{k_E^2}\Big)\\ &+ \log\Big(1 + \frac{m^2 - 2e^2\bar{\phi}^2 + \frac{1}{6}\lambda^2\bar{\phi}^2}{k_E^2} - \frac{e^4(\bar{\phi}^2)^2}{k_E^4}\Big) - 2\log\Big(1 - \frac{e^2\bar{\phi}^2}{k_E^2}\Big)\Big] + \mathrm{const},\end{aligned} \tag{2-96}$$

与上一小节中几何有效势的结果完全相同。

第 3 章　引力修正

在本章中，我们将前文介绍的Vilkovisky-DeWitt方法应用于经引力修正过的标准模型中。

有三种极为重要的情况：即phi-4理论、量子电动力学(QED)，以及Yang-Mills规范理论。这三种理论不仅是量子场论的典型模型，同时也是构建的基本模型的基本组分。本章，我们将通过最小耦合引入这三种模型与引力的相互作用，并研究引力修正的后果。在此过程中，我们使用Vilkovisky-DeWitt方法以保证规范不变性。

3.1　一般讨论

在进入具体计算之前，我们首先指出将Vilkovisky-DeWitt方法应用于引力理论时的诸要点。

3.1.1　引力理论的几何化场空间

依据Vilkovisky-DeWitt方法的精神，场空间的联络系数可由度规诱导而得。因此在处理引力理论时，我们首先需要为场空间引入度规。DeWitt指出，对Einstein引力而言，存在唯一一组含无量纲单参数的度规，满足规范理论的Killing方程组，同时不引入多余的有量纲参量[21]。即：

$$G_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} = \frac{1}{\kappa^2}\sqrt{-g(x)}[g^{\mu(\rho}(x)g^{\sigma)\nu}(x) + \frac{c}{2}g^{\mu\nu}(x)g^{\rho\sigma}(x)]\delta(x-y). \tag{3-1}$$

跟随Toms[15]，我们称之为DeWitt度规。其中c是无量纲参量。

继而，可由此度规计算Christoffel联络系数Γ^k_{ij}，结果为：

$$\begin{aligned}\Gamma^{g_{\lambda\tau}(z)}_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} =&[\frac{1}{4}(g^{\mu\nu}\delta^{\rho}_{(\lambda}\delta^{\sigma}_{\tau)} + g^{\rho\sigma}\delta^{\mu}_{(\lambda}\delta^{\nu}_{\tau)}) - \delta^{(\mu}_{(\lambda}g^{\nu)(\rho}\delta^{\sigma)}_{\tau)} \\ &- \frac{c}{4(2+dc)}g_{\lambda\tau}g^{\mu\nu}g^{\rho\sigma} - \frac{1}{2(2+dc)}g_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu}] \\ &\times\delta(z-x)\delta(z-y).\end{aligned} \tag{3-2}$$

计算细节可见附录。

当引力与物质场相互耦合时，我们尚需将以上度规的定义从单纯的引力场扩展到整个场空间。如前所述，本章考虑三类模型：phi-4模型、量子电动力学，与Yang-Mills规范理论。因此，需要分别为这三种理论定义场空间之度规，并计算出相应的联络。同样，我们在这里列出结果，其计算过程可见附录。

情形1：标量理论 平直时空中的标量phi-4理论由以下Lagrangian给出：

$$\mathcal{L}_\phi = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4. \tag{3-3}$$

当引力进入到理论中后，Lagrangian需被改写成广义坐标协变的形式，即：

$$\mathcal{L}_{\phi(\mathrm{cov})} = \sqrt{-g}[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4]. \tag{3-4}$$

此时，可选取这样的度规，其非零分量由下列诸式给出：

$$G_{g_{\mu\nu}(x),g_{\rho\sigma}(y)}[g] = \tfrac{1}{\kappa^2}\sqrt{-g(x)}[g^{\mu(\rho}(x)g^{\sigma)\nu}(x) + \tfrac{c}{2}g^{\mu\nu}(x)g^{\rho\sigma}(x)]\delta(x-y), \tag{3-5}$$

$$G_{\phi(x)\phi(y)}[g,\phi] = \sqrt{-g(x)}\delta(x-y). \tag{3-6}$$

从而，由该度规所导致Christoffel联络的非零分量为：

$$\begin{aligned}\Gamma^{g_{\lambda\kappa}(z)}_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} &= [\tfrac{1}{4}(\eta^{\mu\nu}\delta^\rho_{(\lambda}\delta^\sigma_{\kappa)} + \eta^{\rho\sigma}\delta^\mu_{(\lambda}\delta^\nu_{\kappa)}) - \delta^{(\mu}_{(\lambda}\eta^{\nu)(\rho}\delta^{\sigma)}_{\kappa)} \\ &\quad - \tfrac{1}{4(d-2)}\eta_{\lambda\kappa}\eta^{\mu\nu}\eta^{\rho\sigma} + \tfrac{1}{2(d-2)}\eta_{\lambda\kappa}\eta^{\mu(\rho}\eta^{\sigma)\nu}] \\ &\quad \times\delta(z-x)\delta(z-y).\end{aligned} \tag{3-7}$$

$$\Gamma^{\phi(z)}_{g_{\mu\nu}(x)\phi(y)} = \tfrac{1}{4}\eta^{\mu\nu}(z)\delta(z-x)\delta(z-y). \tag{3-8}$$

$$\Gamma^{g_{\lambda\kappa}(z)}_{\phi(x)\phi(y)} = \tfrac{1}{2(d-2)}\kappa^2\eta_{\lambda\kappa}(z)\delta(z-x)\delta(z-y). \tag{3-9}$$

情形2: $SU(N)$规范理论 平直空间的纯$SU(N)$规范理论，或Yang-Mills理论，由以下Lagrangian给出：

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}, \tag{3-10}$$

同样，当引力进入后，广义坐标协变的Lagrangian为：

$$\mathcal{L} = -\tfrac{1}{4}\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}. \tag{3-11}$$

此模型可被称为Einstein-Yang-Mills理论。与此类似，引力修正的量子电动力学可被称作Einstein-Maxwell理论。下面，我们列出Einstein-Yang-Mills理论的

度规与联络的非零分量。从中直接拿去规范指标，即可得到适用于 Einstein-Maxwell理论的结果。

首先，场空间度规的非零分量为：

$$G_{g_{\mu\nu}(x),g_{\rho\sigma}(y)}[g] = \frac{1}{\kappa^2}\sqrt{-g(x)}[g^{\mu(\rho}(x)g^{\sigma)\nu}(x) + \frac{c}{2}g^{\mu\nu}(x)g^{\rho\sigma}(x)]\delta(x-y), \tag{3-12}$$

$$G_{A^a_\mu(x)A^b_\nu(y)} = \sqrt{-g(x)}\delta^{ab}g^{\mu\nu}(x)\delta(x-y). \tag{3-13}$$

由此可得Christoffel联络的非零分量为：

$$\begin{aligned}\Gamma^{g_{\lambda\kappa}(z)}_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} = &\Big[\frac{1}{4}(\eta^{\mu\nu}\delta^{\rho}_{(\lambda}\delta^{\sigma}_{\kappa)} + \eta^{\rho\sigma}\delta^{\mu}_{(\lambda}\delta^{\nu}_{\kappa)}) - \delta^{(\mu}_{(\lambda}\eta^{\nu)(\rho}\delta^{\sigma)}_{\kappa)} \\ &- \frac{1}{4(d-2)}\eta_{\lambda\kappa}\eta^{\mu\nu}\eta^{\rho\sigma} + \frac{1}{2(d-2)}\eta_{\lambda\kappa}\eta^{\mu(\rho}\eta^{\sigma)\nu}\Big] \\ &\times\delta(z-x)\delta(z-y);\end{aligned} \tag{3-14}$$

$$\Gamma^{A^c_\lambda(z)}_{g_{\mu\nu}(x)A^b_\rho(y)} = \frac{1}{4}[g^{\mu\nu}(z)\delta^{\rho}_{\lambda} - g^{\rho\mu}(z)\delta^{\nu}_{\alpha} - g^{\rho\nu}(z)\delta^{\mu}_{\lambda}]\delta^{bc}\delta(z-x)\delta(z-y); \tag{3-15}$$

$$\Gamma^{g_{\lambda\kappa}(z)}_{\phi(x)\phi(y)} = \frac{\kappa^2}{2}[\delta^{(\mu}_{\lambda}\delta^{\nu)}_{\kappa} - \frac{1+c}{2+dc}g_{\lambda\kappa}(x)g^{\mu\nu}(x)]\delta^{ab}\delta(z-x)\delta(z-y). \tag{3-16}$$

3.2 标量理论

在我们的模型中，共有三类Lagrangian将贡献到进行圈图计算所需的Feynman规则中。第一类为经典理论的Lagrangian，第二类由规范固定产生，包括规范固定项与鬼项。第三类由Vilkovisky-DeWitt方法所引入，它们来自场空间的协变泛函微商中的联络，因此我们称之为联络项。

在接下来三小节中，我们将逐步给出这三类Lagrangian项的形式。

3.2.1 经典Lagrangian

如上提及，平直空间中的phi-4理论可由以下Lagrangian给出：

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4. \tag{3-17}$$

当引力进入后，广义坐标协变的Lagrangian为：

$$\mathcal{L}_{\phi(\mathrm{cov})} = \sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4]. \tag{3-18}$$

将度规在平直的Minkowski背景$\eta_{\mu\nu}$附近作展开：

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{3-19}$$

则以上Lagrangian可被相应地展开为：

$$\begin{aligned}\mathcal{L}_{\phi(\mathrm{cov})} =&[1+\tfrac{1}{2}\kappa h+\tfrac{1}{8}\kappa^2(h^2-2h_{\mu\nu}h^{\mu\nu})]\\&\times[\tfrac{1}{2}(\eta^{\mu\nu}-\kappa h^{\mu\nu}+\kappa^2 h^{\mu}_{\lambda}h^{\nu\lambda})\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}m^2\phi^2-\tfrac{1}{4!}\lambda\phi^4]+O(\kappa^3),\end{aligned}$$

按展开系数κ的幂次重新排列各项，可得，

$$\begin{aligned}\mathcal{L}_{\phi(\mathrm{cov})} =&\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m^2\phi^2-\tfrac{1}{4!}\lambda\phi^4\\&+\tfrac{1}{2}\kappa[-h_{\mu\nu}\partial^\mu\phi\partial^\nu\phi+\tfrac{1}{2}h\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m^2h\phi^2-\tfrac{1}{4!}\lambda h\phi^4]\\&+\tfrac{1}{8}\kappa^2(h^2-2h_{\mu\nu}h^{\mu\nu})[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m^2\phi^2-\tfrac{1}{4!}\lambda\phi^4]\\&+\tfrac{1}{2}\kappa^2h^{\lambda}_{\mu}h_{\nu\lambda}\partial^\mu\phi\partial^\nu\phi-\tfrac{1}{4}\kappa^2hh_{\mu\nu}\partial^\mu\phi\partial^\nu\phi+O(\kappa^3).\end{aligned} \tag{3-20}$$

根据有效作用量方法的精神，我们同样对标量ϕ在其背景值$\bar{\phi}(x)$附近作展开：

$$\phi(x)=\bar{\phi}(x)+\varphi(x). \tag{3-21}$$

此后，$\eta_{\mu\nu}$和$\bar{\phi}(x)$将被视为经典的背景场，而$h_{\mu\nu}(x)$与$\varphi(x)$代表量子涨落，我们需要对它们作量子化。

将以上展开代入原Lagrangian，并且保留到量子涨落（$h_{\mu\nu}$和φ）的二次项，我们得到：

$$\mathcal{L}_\varphi=\tfrac{1}{2}\varphi[-\partial^2-m^2-\tfrac{1}{2}\lambda\bar{\phi}^2]\varphi \tag{3-22}$$

$$+\tfrac{1}{2}\kappa[-2h_{\mu\nu}\partial^\mu\bar{\phi}\partial^\nu\varphi+h\partial_\mu\bar{\phi}\partial^\mu\varphi-m^2\bar{\phi}h\varphi-\tfrac{1}{6}\lambda\bar{\phi}^3h\varphi] \tag{3-23}$$

$$\begin{aligned}&+\tfrac{1}{8}\kappa^2(h^2-2h_{\mu\nu}h^{\mu\nu})\bar{\mathcal{L}}_0+\tfrac{1}{2}\kappa^2h^{\lambda}_{\mu}h_{\nu\lambda}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi}\\&-\tfrac{1}{4}\kappa^2hh_{\mu\nu}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi}.\end{aligned} \tag{3-24}$$

同时，我们也须加入由Hilbert-Einstein作用量贡献的项，

$$\mathcal{L}_h=\tfrac{1}{4}[h\partial^2h-h^{\mu\nu}\partial^2h_{\mu\nu}+2h_{\mu\lambda}\partial^\mu\partial_\nu h^{\nu\lambda}-2h\partial^\mu\partial^\nu h_{\mu\nu}]+O(\kappa), \tag{3-25}$$

以及宇宙学常数项，

$$\mathcal{L}_\Lambda=-\frac{\Lambda}{\kappa}h-\frac{\Lambda}{4}(h^2-2h_{\mu\nu}h^{\mu\nu})+O(\kappa). \tag{3-26}$$

3.2.2 Landau-DeWitt规范

前文已提到，在Landau-DeWitt规范下计算将得到极大的简化。我们在本小节中找出此规范的具体形式。由定义，Landau-DeWitt规范可从场量规范变换中

读出。因此，从场量的规范变换开始：

$$\delta g_{\mu\nu} = -\epsilon^{\lambda}\partial_{\lambda} g_{\mu\nu} - g_{\lambda\nu}\partial_{\mu}\epsilon^{\lambda} - g_{\mu\lambda}\partial_{\nu}\epsilon^{\lambda}, \tag{3-27}$$

$$\delta\phi = -\epsilon^{\lambda}\partial_{\lambda}\phi. \tag{3-28}$$

由此可以导出度规之涨落的规范变换$h_{\mu\nu}$，为：

$$\delta h_{\mu\nu} = -\frac{1}{\kappa}(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}) - (\epsilon^{\lambda}\partial_{\lambda}h_{\mu\nu} + h_{\lambda\nu}\partial_{\mu}\epsilon^{\lambda} + h_{\mu\lambda}\partial_{\nu}\epsilon^{\lambda}). \tag{3-29}$$

此时，我们将参数ϵ^{λ}重新标度化为$\epsilon^{\lambda}/\kappa$以使之成为无量纲量。从而，规范变换可被修改为：

$$\delta h_{\mu\nu} = -\partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} - \kappa(\epsilon^{\lambda}\partial_{\lambda}h_{\mu\nu} + h_{\lambda\nu}\partial_{\mu}\epsilon^{\lambda} + h_{\mu\lambda}\partial_{\nu}\epsilon^{\lambda}), \tag{3-30}$$

$$\delta\phi = -\kappa\epsilon^{\lambda}\partial_{\lambda}\phi. \tag{3-31}$$

从而可读出规范变换的生成元，为：

$$K^{g_{\mu\nu}(x)}_{\lambda y}[g] = -\kappa[\partial_{\lambda}g_{\mu\nu}(x) + g_{\lambda\nu}(x)\partial_{\mu} + g_{\mu\lambda}(x)\partial_{\nu}]\delta(x-y), \tag{3-32}$$

$$K^{\phi(x)}_{\lambda y}[\phi] = -\kappa\partial_{\lambda}\phi(x)\delta(x-y). \tag{3-33}$$

我们需要找到的规范固定项为

$$\mathcal{L}_{\mathrm{GF}} = \frac{1}{2\alpha}F[h_{\mu\nu},\varphi]^2, \tag{3-34}$$

其中F是规范固定条件。根据Landau-DeWitt规范之定义，它是：

$$\begin{aligned} F[h_{\mu\nu},\varphi] =& \frac{1}{2}\int \mathrm{d}^d y \mathrm{d}^d z\,[K^{g_{\mu\nu}(y)}_{\lambda x} G_{g_{\mu\nu}(y)g_{\rho\sigma}(z)}\kappa h_{\rho\sigma}(z) \\ &+ K^{\phi(y)}_{\lambda x} G_{\phi(y)\phi(z)}\varphi(z)]. \end{aligned} \tag{3-35}$$

其中，K与G应当取背景值。由于我们已经知道了度规G_{ij}的具体形式，故而，

$$\begin{aligned} &\frac{1}{2}\int \mathrm{d}^d y \mathrm{d}^d z\, K^{g_{\mu\nu}(y)}_{\lambda x}[\eta] G_{g_{\mu\nu}(y)g_{\rho\sigma(z)}}[\eta]\kappa h_{\rho\sigma}(z) \\ =& -\frac{1}{2}\int \mathrm{d}^d y \mathrm{d}^d z\,[\eta_{\lambda\nu}(x)\partial_{\mu} + \eta_{\mu\lambda}(x)\partial_{\nu}]\delta(x-y) \\ &\times [\eta^{\mu(\rho}\eta^{\sigma)\nu} + \frac{c}{2}\eta^{\mu\nu}\eta^{\rho\sigma}]\delta(y-z)h_{\rho\sigma}(z) \\ =& \partial^{\mu}h_{\mu\lambda} + \frac{c}{2}\partial_{\lambda}h \end{aligned}$$

以及，

$$\begin{aligned}
&\frac{1}{2}\int \mathrm{d}^d y \mathrm{d}^d z\, K_{\lambda x}^{\phi(y)}[\bar{\phi}] G_{\phi(y)\phi(z)}[\eta,\bar{\phi}]\varphi(z)\\
=&-\frac{1}{2}\int \mathrm{d}^d y \mathrm{d}^d z\, \kappa[\partial_\lambda \bar{\phi}(x)]\delta(x-y)\delta(y-z)\varphi(z)\\
=&\ -\tfrac{\kappa}{2}[\partial_\lambda \bar{\phi}(x)]\varphi(x).
\end{aligned}$$

因此，规范固定条件为：

$$F_\lambda[h_{\mu\nu},\varphi] = \partial^\mu h_{\mu\lambda} + \tfrac{c}{2}\partial_\lambda h - \tfrac{1}{2}\kappa(\partial_\lambda\bar{\phi})\varphi, \tag{3-36}$$

由此可写出Lagrangian中的规范固定项：

$$\begin{aligned}
\tfrac{1}{2\alpha}F^2 = &\tfrac{1}{2\alpha}(\partial^\mu h_{\mu\lambda} + \tfrac{c}{2}\partial_\mu h)^2 + \tfrac{1}{8\alpha}\kappa^2(\partial_\lambda\bar{\phi})^2\varphi^2\\
&- \tfrac{1}{2\alpha}\kappa\partial^\lambda\bar{\phi}(\partial^\mu h_{\mu\lambda} + \tfrac{c}{2}\partial_\lambda h)\varphi.
\end{aligned} \tag{3-37}$$

其中，c即为出现在场空间度规中的参量。可见，若取之为1，则对引力$h_{\mu\nu}$部分的规范固定即取常见的谐和规范形式。在此规范下，引力子的传播子将得到简化。在$c = 1$时：

$$F_\lambda[h_{\mu\nu},\varphi] = \partial^\mu h_{\mu\lambda} - \tfrac{1}{2}\partial_\lambda h - \tfrac{1}{2}\kappa(\partial_\lambda\bar{\phi})\varphi, \tag{3-38}$$

and

$$\begin{aligned}
\tfrac{1}{2\alpha}F^2 = &\tfrac{1}{2\alpha}(\partial^\mu h_{\mu\lambda} - \tfrac{1}{2}\partial_\mu h)^2 + \tfrac{1}{8\alpha}\kappa^2(\partial_\lambda\bar{\phi})^2\varphi^2\\
&- \tfrac{1}{2\alpha}\kappa\partial^\lambda\bar{\phi}(\partial^\mu h_{\mu\lambda} - \tfrac{1}{2}\partial_\lambda h)\varphi.
\end{aligned} \tag{3-39}$$

需要指出，为了得到正确的规范无关的结果，我们应当在计算的末尾取$\alpha \to 0$的条件。

鬼项 Landau-DeWitt规范将引入非平凡的鬼项。即：

$$\mathcal{L}_{\text{ghost}} = \bar{c}^\alpha \frac{\delta F_\alpha[h_{\mu\nu},\varphi]}{\delta\epsilon^\beta} c^\beta. \tag{3-40}$$

其中，

$$\frac{\delta F_\alpha[h_{\mu\nu},\varphi]}{\delta\epsilon^\beta} = \frac{\delta F_\alpha[h_{\mu\nu},\varphi]}{\delta(\kappa h_{\mu\nu})}K_\beta^{g_{\mu\nu}}[\eta] + \frac{\delta F_\alpha[h_{\mu\nu},\varphi]}{\delta\varphi}K_\beta^{\phi}[\eta,\bar{\phi}]. \tag{3-41}$$

因此，由(3-36), (3-32)以及(3-33)三式，可得

$$\begin{aligned}\frac{\delta F_\alpha[h_{\mu\nu},\varphi]}{\delta\epsilon^\beta} =&\frac{1}{\kappa}[\delta_\lambda^{(\mu}\partial^{\nu)}+\frac{c}{2}\eta^{\mu\nu}\partial_\alpha]\kappa[\eta_{\beta\nu}\partial_\mu+\eta_{\mu\beta}\partial_\nu]+\frac{1}{2}\kappa^2(\partial_\alpha\bar{\phi})(\partial_\beta\bar{\phi})\\ =&-\eta_{\alpha\beta}\partial^2-(1+c)\partial_\mu\partial_\nu+\frac{1}{2}\kappa^2(\partial_\alpha\bar{\phi})(\partial_\beta\bar{\phi}).\end{aligned} \tag{3-42}$$

因而，我们得到鬼项，为：

$$\mathcal{L}_{\text{ghost}}=-\bar{c}^\mu[\eta_{\mu\nu}\partial^2+(1+c)\partial_\mu\partial_\nu]c^\nu+\frac{1}{2}\kappa^2\bar{c}^\mu(\partial_\mu\bar{\phi})(\partial_\nu\bar{\phi})c^\nu. \tag{3-43}$$

3.2.3 联络项

联络项来自场空间的协变泛函微商，即：

$$\nabla_i\nabla_j S=\frac{\delta^2 S}{\delta\phi_i\delta\phi_j}-\Gamma^{\phi_k}_{\phi_i\phi_j}\frac{\delta S}{\delta\phi_k}. \tag{3-44}$$

容易看出，联络项完全来自上式右端第二项的贡献。因此需要计算作用量对场量的一阶泛函微商在背景场上的取值，即：

$$\frac{\delta S_\phi}{\delta(\kappa h_{\lambda\kappa})}=-\frac{1}{2}\partial^\lambda\bar{\phi}\partial^\kappa\bar{\phi}+\frac{1}{2}\eta^{\lambda\kappa}\bar{\mathcal{L}}_0, \tag{3-45}$$

$$\frac{\delta S_h}{\delta(\kappa h_{\lambda\kappa})}=0, \tag{3-46}$$

$$\frac{\delta S_\Lambda}{\delta(\kappa h_{\lambda\kappa})}=-\frac{\Lambda}{\kappa^2}\eta^{\lambda\kappa}, \tag{3-47}$$

$$\frac{\delta S_\phi}{\delta\phi}=-\partial^2\phi-m^2\bar{\phi}-\frac{1}{6}\lambda\bar{\phi}^3. \tag{3-48}$$

另外，我们已在此前给出了所需的联络系数。故此时可立即写出联络项，为：

$$\begin{aligned}&-\frac{1}{2}\int \mathrm{d}^d z\mathrm{d}^d u\mathrm{d}^d w\,\Gamma^{g_{\lambda\kappa}(z)}_{g_{\mu\nu}(u)g_{\rho\sigma}(w)}\frac{\delta\mathcal{L}(x)}{\delta g_{\lambda\kappa}(z)}\kappa^2 h_{\mu\nu}(u)h_{\rho\sigma}(w).\\ =&\frac{1}{8}\kappa^2 hh_{\mu\nu}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi}-\frac{1}{4}\kappa^2 h^\lambda_\mu h_{\lambda\nu}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi}\\ &-\frac{1}{8(d-2)}[\frac{1}{2}\kappa^2(\partial\bar{\phi})^2+\frac{d-4}{2}\kappa^2\bar{\mathcal{L}}_0-(d-4)\Lambda](h^2-2h_{\mu\nu}h^{\mu\nu}).\end{aligned} \tag{3-49}$$

$$\begin{aligned}&-\int \mathrm{d}^d z\mathrm{d}^d u\mathrm{d}^d w\,\Gamma^{\phi(z)}_{g_{\mu\nu}(u)\phi(w)}\frac{\delta\mathcal{L}(x)}{\delta\phi(z)}\kappa h_{\mu\nu}(u)\varphi(w)\\ =&\frac{\kappa}{4}[\partial^2\bar{\phi}+m^2\bar{\phi}+\frac{1}{6}\lambda\bar{\phi}^3]h\varphi.\end{aligned} \tag{3-50}$$

$$-\frac{1}{2}\int \mathrm{d}^d z\mathrm{d}^d u\mathrm{d}^d w\,\Gamma^{g_{\lambda\kappa}(z)}_{\phi(u)\phi(w)}\frac{\delta\mathcal{L}(x)}{\delta g_{\lambda\kappa}(z)}\varphi(u)\varphi(w)$$

$$= \frac{1}{4(d-2)}\kappa^2[\tfrac{1}{2}(\partial\bar{\phi})^2 - \tfrac{d}{2}\bar{\mathcal{L}}_0 + \tfrac{d}{\kappa^2}\Lambda]\varphi^2. \tag{3-51}$$

3.2.4 Feynman规则

我们在此总结以上三小节的结果。首先，经典Lagrangian展开至κ^2的结果为：

$$\begin{aligned}
\mathcal{L}_\phi =& \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4 \\
&+ \tfrac{1}{2}\kappa[-h_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + \tfrac{1}{2}h\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2h\phi^2 - \tfrac{1}{4!}\lambda h\phi^4] \\
&+ \tfrac{1}{8}\kappa^2(h^2 - 2h_{\mu\nu}h^{\mu\nu})[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4] \\
&+ \tfrac{1}{2}\kappa^2h_\mu^\lambda h_{\nu\lambda}\partial^\mu\phi\partial^\nu\phi - \tfrac{1}{4}\kappa^2hh_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + O(\kappa^3).
\end{aligned} \tag{3-52}$$

从中取出关于量子涨落场的二阶项，即有：

$$\begin{aligned}
\mathcal{L}_\varphi =& \tfrac{1}{2}\varphi[-\partial^2 - m^2 - \tfrac{1}{2}\lambda\bar{\phi}^2]\varphi \\
&+ \tfrac{1}{2}\kappa[-2h_{\mu\nu}\partial^\mu\bar{\phi}\partial^\nu\varphi + h\partial_\mu\bar{\phi}\partial^\mu\varphi - m^2\bar{\phi}h\varphi - \tfrac{1}{6}\lambda\bar{\phi}^3h\varphi] \\
&+ \tfrac{1}{8}\kappa^2(h^2 - 2h_{\mu\nu}h^{\mu\nu})\bar{\mathcal{L}}_0 + \tfrac{1}{2}\kappa^2h_\mu^\lambda h_{\nu\lambda}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi} \\
&- \tfrac{1}{4}\kappa^2hh_{\mu\nu}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi}.
\end{aligned}$$

其次，我们有来自Hilbert-Einstein作用量的项，它贡献引力子的传播子：

$$\mathcal{L}_h = \tfrac{1}{4}[h\partial^2h - h^{\mu\nu}\partial^2h_{\mu\nu} + 2h_{\mu\lambda}\partial^\mu\partial_\nu h^{\nu\lambda} - 2h\partial^\mu\partial^\nu h_{\mu\nu}] + O(\kappa). \tag{3-54}$$

再次，我们有宇宙学项：

$$\mathcal{L}_\Lambda = -\frac{\Lambda}{\kappa}h - \frac{\Lambda}{4}(h^2 - 2h_{\mu\nu}h^{\mu\nu}) + O(\kappa). \tag{3-55}$$

另外，对理论作量子化将引入如下的规范固定项

$$\begin{aligned}
\mathcal{L}_{\mathrm{GF}} =& \tfrac{1}{2\alpha}(\partial^\mu h_{\mu\lambda} + \tfrac{c}{2}\partial_\mu h)^2 + \tfrac{1}{8\alpha}\kappa^2(\partial_\lambda\bar{\phi})^2\varphi^2 \\
&- \tfrac{1}{2\alpha}\kappa\partial^\lambda\bar{\phi}(\partial^\mu h_{\mu\lambda} + \tfrac{c}{2}\partial_\lambda h)\varphi.
\end{aligned} \tag{3-56}$$

以及鬼项：

$$\mathcal{L}_{\mathrm{ghost}} = -\bar{c}^\mu[\eta_{\mu\nu}\partial^2 + (1+c)\partial_\mu\partial_\nu]c^\nu + \tfrac{1}{2}\kappa^2\bar{c}^\mu(\partial_\mu\bar{\phi})(\partial_\nu\bar{\phi})c^\nu. \tag{3-57}$$

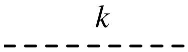
最后，我们还有联络项：

$$\mathcal{L}_{\mathrm{conn.}} = \tfrac{1}{8}\kappa^2hh_{\mu\nu}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi} - \tfrac{1}{4}\kappa^2h_\mu^\lambda h_{\lambda\nu}\partial^\mu\bar{\phi}\partial^\nu\bar{\phi}$$

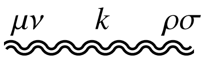
$$
\begin{aligned}
&-\frac{1}{8(d-2)}\Big[\frac{1}{2}\kappa^2(\partial\bar{\phi})^2+\frac{d-4}{2}\kappa^2\bar{\mathcal{L}}_0-(d-4)\Lambda\Big](h^2-2h_{\mu\nu}h^{\mu\nu})\\
&+\frac{\kappa}{4}[\partial^2\bar{\phi}+m^2\bar{\phi}+\frac{1}{6}\lambda\bar{\phi}^3]h\varphi+\frac{1}{4(d-2)}\kappa^2\Big[\frac{1}{2}(\partial\bar{\phi})^2-\frac{d}{2}\bar{\mathcal{L}}_0+\frac{d}{\kappa^2}\Lambda\Big]\varphi^2. \qquad (3\text{-}58)
\end{aligned}
$$

至此，该理论的Feynman规则即可从以上各项中读出，略去平庸但冗长的计算，我们在此列出结果如下。

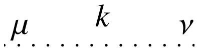
传播子

φ : 

$$= \frac{i}{k^2-m^2+\frac{d}{2(d-2)}\Lambda}$$

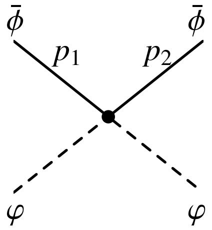
$h_{\mu\nu}$: 

$$= \frac{i}{k^2+(2-\frac{d-4}{d-2})\Lambda}\Big[(2\eta_{\mu(\rho}\eta_{\sigma)\nu}-\frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}) - (1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)}}{k^2+(2-\frac{d-4}{d-2})\alpha\Lambda}\Big].$$

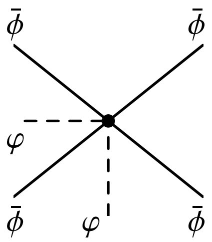
c : 

$$= \frac{i}{k^2}\eta_{\mu\nu}$$

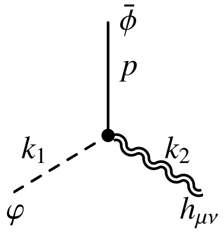
顶角



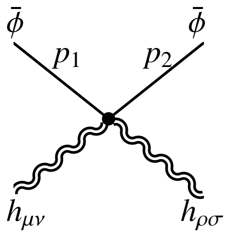
$$= -i\lambda + i\kappa^2\Big[\frac{1}{4}-\frac{1}{2\alpha}\Big]p_1\cdot p_2 + i\kappa^2\frac{d}{4(d-2)}m^2.$$



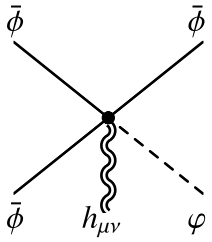
$$= i\kappa^2\lambda\frac{d}{4(d-2)}$$



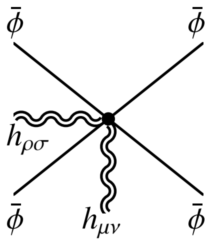
$$=i\kappa k_1^{(\mu}p^{\nu)} + \frac{i}{2\alpha}\kappa p^{(\mu}k_2^{\nu)} - \frac{i}{2}\kappa[k_1 \cdot p + \frac{1}{2}p^2 - \frac{1}{2\alpha}p \cdot k_2 + \frac{1}{2}m^2]\eta^{\mu\nu}$$



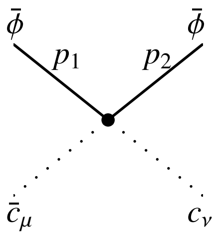
$$=\frac{i}{8}\kappa^2[p_1 \cdot p_2 - \frac{d}{d-2}m^2][\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu}] + \frac{i}{4}\kappa^2[\eta^{\mu\nu}p_1^{(\rho}p_2^{\sigma)} + \eta^{\rho\sigma}p_1^{(\mu}p_2^{\nu)}] - i\kappa^2 p_1^{(\mu}\eta^{\nu)(\rho}p_2^{\sigma)}.$$



$$= -\frac{i}{4}\kappa\lambda\eta^{\mu\nu}$$



$$= -i\kappa^2\lambda\frac{d}{8(d-2)}[\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu}]$$



$$= -i\kappa^2 p_1^\mu p_2^\nu$$

注意我们在此已取场空间度规中的任意参数$c = -1$以简化引力子的传播子。在任意c的情形下，引力子的传播子将变得异常复杂。

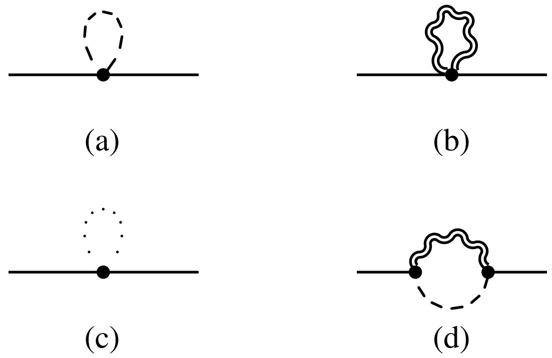


图 3.1　标量自能一圈图

3.2.5　标量自能

现在，我们已经有Feynman规则在手以计算所需量。首先是标量场的自能，即两点顶角函数。此量之所以重要，是因为我们可以从中读出标量的波函数重整化，而后者将贡献至理论的β函数中。基于同样的理由，我们亦无必要计算完整的顶角函数，而只需算出正比于外动量p^2的项即可。因为，只有这样的项贡献进标量的波函数重整化。

标量自能的一圈图如图3.1所示。 在对此处及以下的圈图进行计算时，我们使用了截断正规化与最小剪除手续。将各图的详细计算移至附录，我们在此只列出结果如下。这里的结果只包含独立于外动量p以及正比于p^2的项。p_μ的高阶项对我们并不重要。

$$(a) = \frac{i}{32\pi^2}[\lambda + \kappa^2(\tfrac{1}{4} - \tfrac{1}{2\alpha})p^2 - \tfrac{1}{2}\kappa^2 m^2][-\mu^2 + (m^2 - \Lambda)\log\mu^2]; \tag{3-59}$$

$$(b) = -\frac{3i}{16\pi^2}\kappa^2 m^2[-\mu^2 - 2\Lambda\log\mu^2]; \tag{3-60}$$

$$(c) = -\frac{i}{16\pi^2}\kappa^2 p^2\mu^2; \tag{3-61}$$

$$\begin{aligned}(d) = &\frac{\mathrm{i}\kappa^2}{16\pi^2}(1 - \tfrac{1}{4\alpha})p^2\mu^2 \\ &+ \frac{\mathrm{i}\kappa^2}{16\pi^2}[\tfrac{1}{4\alpha}p^2(m^2 - \Lambda) - (\tfrac{3}{4}m^2 - \tfrac{1}{2}\Lambda)p^2 - \tfrac{3}{4}m^4]\log\mu^2.\end{aligned} \tag{3-62}$$

综上，可得自能一圈图部分中有关的项，为：

$$\begin{aligned}\Gamma^{(2)}(p) = &\frac{\mathrm{i}}{128\pi^2}\Big[-4\lambda + 26\kappa^2 m^2 - \kappa^2 p^2\Big]\mu^2 \\ &+ \frac{\mathrm{i}}{128\pi^2}\Big[\kappa^2(3\Lambda - 5m^2)p^2 - 8m^4\kappa^2 - 4\lambda\Lambda + 4m^2\lambda + 50\kappa^2 m^2\Lambda\Big]\log\mu^2.\end{aligned} \tag{3-63}$$

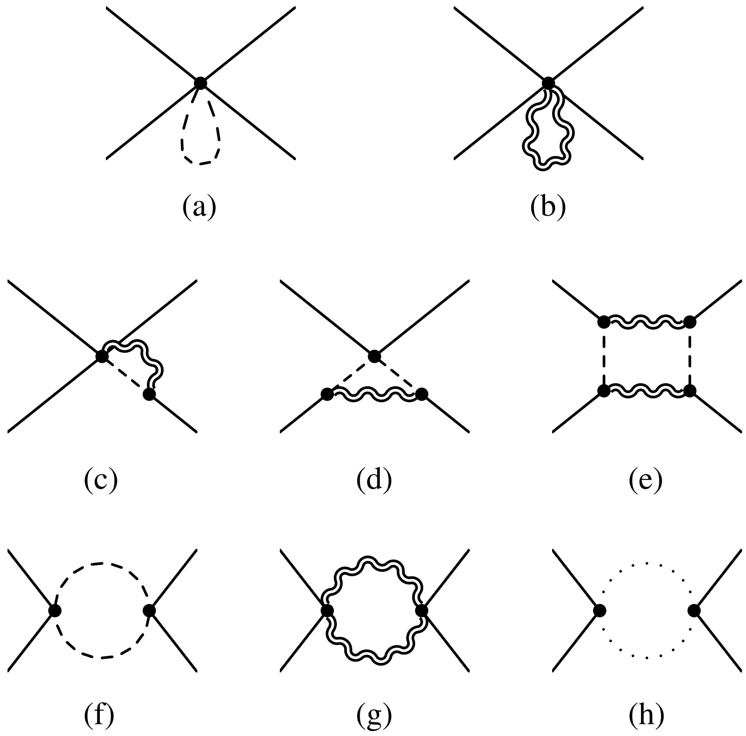


图 3.2　标量四点顶角函数一圈图

3.2.6　四点正规顶角函数

通过四点顶角函数，可以计算出耦合常数的量子修正。四点顶角函数的一圈图如图3.2所示。 由于我们只关心对$\lambda\phi^4$类型顶角的修正，因此可以取所有外动量$p = 0$以简化计算。不难看出，(d)、(e)、(g)以及 (h)在零外动量情形下将无发散。从而我们只需计算(a)~(c), (f)和(g)各图。

注意到，我们尚需将(f)与(g)的结果乘以3，因为每张图皆代表三个通道的贡献。我们也需将(c)图的结果乘以4，因为其中的圈可落于四条外线的任一条之上。

以下是贡献发散的各图的结果。其推导可见附录。

$$\text{(a)} = \frac{i}{64\pi^2}\kappa^2\lambda[\mu^2 - (m^2 - \tfrac{1}{2}\Lambda)\log\mu^2]; \tag{3-64}$$

$$\text{(b)} = \frac{3i}{16\pi^2}\kappa^2\lambda[\mu^2 + 2\Lambda\log\mu^2]. \tag{3-65}$$

$$\text{(c)} = -\frac{3i}{64\pi^2}\kappa^2\lambda m^2\log\mu^2. \tag{3-66}$$

$$\text{(f)} = \frac{i}{32\pi^2}[\lambda - \tfrac{1}{2}\kappa^2 m^2]^2\log\mu^2. \tag{3-67}$$

$$\text{(g)} = \frac{3i}{16\pi^2}\kappa^4 m^4\log\mu^2. \tag{3-68}$$

由以上计算，我们发现四点顶角函数中所需的项在一圈图水平为：

$$\Gamma^{(4)}(p_i=0)=-i\lambda+\tfrac{13}{64\pi^2}i\lambda\kappa^2\mu^2 + \tfrac{i}{64\pi^2}\left[-19\kappa^2\lambda m^2+\tfrac{49}{2}\kappa^2\lambda\Lambda+6\lambda^2+\tfrac{75}{2}\kappa^4 m^4\right]\log\mu^2. \tag{3-69}$$

3.2.7 β函数

不难从以上结果中读出引力效应对β函数的修正。我们在此只考虑领头项，即二次发散。此时，有：

$$Z_\lambda=1-\tfrac{13}{64\pi^2}\kappa^2(M^2-\mu^2); \tag{3-70}$$

$$Z_\phi=1+\tfrac{1}{128\pi^2}\kappa^2(M^2-\mu^2). \tag{3-71}$$

因此，

$$\beta(\lambda)=\frac{\partial\lambda}{\partial\log\mu}=\frac{3\lambda\kappa^2\mu^2}{8\pi^2}. \tag{3-72}$$

即，此时并无渐进自由。

3.3 Einstein-Maxwell理论与Einstein-Yang-Mills理论

完全平行于对标量理论的处理，此时对Feynman规则有贡献的Lagrangian亦来自三部分：经典Lagrangian、规范固定，与联络项。以下我们分别处理之。

3.3.1 经典理论

平直背景下纯$U(1)$规范场的Lagrangian由下式给出：

$$\mathcal{L}_{\mathrm{QED}}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3-73}$$

在引力进入后，广义坐标协变的Lagrangian为：

$$\mathcal{L}_{\mathrm{E\text{-}M}}=-\tfrac{1}{4}\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{3-74}$$

其中，$F_{\mu\nu}$为电磁场的场强张量：

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{3-75}$$

一如既往，对度规$g_{\mu\nu}$作如下展开，

$$g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}, \tag{3-76}$$

以及

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}_{\alpha} h^{\alpha\nu} + O(\kappa^3). \tag{3-77}$$

我们得到：

$$\begin{aligned}\mathcal{L}_{\text{E-M}} = & -\frac{1}{4}[1+\frac{1}{2}\kappa h+\frac{1}{8}\kappa^2(h^2-2h_{\lambda\kappa}h^{\lambda\kappa})] \\ & \times(\eta^{\mu\rho}-\kappa h^{\mu\rho}+\kappa^2 h^{\mu}_{\alpha}h^{\alpha\rho})(\eta^{\nu\sigma}-\kappa h^{\nu\sigma}+\kappa^2 h^{\nu}_{\beta}h^{\beta\sigma})F_{\mu\nu}F_{\rho\sigma}+O(\kappa^3) \\ = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}\kappa[\frac{1}{2}hF_{\mu\nu}F^{\mu\nu}-2h^{\mu\rho}F_{\mu\nu}F_{\rho}{}^{\nu}] \\ & -\frac{1}{4}\kappa^2[\frac{1}{8}(h^2-2h_{\rho\sigma}h^{\rho\sigma})F_{\mu\nu}F^{\mu\nu}+(2h^{\mu}_{\alpha}h^{\alpha\rho}-hh^{\mu\rho})F_{\mu\nu}F_{\rho}{}^{\nu} \\ & +h^{\mu\rho}h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}]+O(\kappa^3).\end{aligned} \tag{3-78}$$

3.3.2 Landau-DeWitt规范

此时，理论的规范变换分为两部分：引力部分的广义坐标变换，以及电磁场部分的$U(1)$规范变换。我们用ϵ_μ对前者进行参数化，用ϵ对后者作参数化。于是，在此两种变换的联合作用下，各场量的变换为：

$$\delta g_{\mu\nu} = -\epsilon^{\lambda}\partial_{\lambda}g_{\mu\nu} - g_{\lambda\nu}\partial_{\mu}\epsilon^{\lambda} - g_{\mu\lambda}\partial_{\nu}\epsilon^{\lambda}, \tag{3-79}$$

$$\delta A_{\mu} = -\epsilon^{\lambda}\partial_{\lambda}A_{\mu} - A_{\lambda}\partial_{\mu}\epsilon^{\lambda} + \partial_{\mu}\epsilon. \tag{3-80}$$

由此，可读出规范变换的生成元为：

$$K^{g_{\mu\nu}(x)}_{\lambda y} = -\kappa[\partial_{\lambda}g_{\mu\nu}(x) + g_{\lambda\nu}(x)\partial_{\mu} + g_{\mu\lambda}(x)\partial_{\nu}]\delta(x-y), \tag{3-81}$$

$$K^{g_{\mu\nu}(x)}_{y} = 0, \tag{3-82}$$

$$K^{A_{\mu}(x)}_{\lambda y} = -\kappa[\partial_{\lambda}A_{\mu}(x) + A_{\lambda}(x)\partial_{\mu}]\delta(x-y), \tag{3-83}$$

$$K^{A_{\mu}(x)}_{y} = \partial_{\mu}\delta(x-y). \tag{3-84}$$

另一方面，我们已经知道了Einstein-Maxwell理论之场空间度规的非零分量，为：

$$G_{g_{\mu\nu}(x),g_{\rho\sigma}(y)}[g] = \frac{1}{\kappa^2}\sqrt{-g(x)}[g^{\mu(\rho}(x)g^{\sigma)\nu}(x) + \frac{c}{2}g^{\mu\nu}(x)g^{\rho\sigma}(x)]\delta(x-y), \tag{3-85}$$

$$G_{A_{\mu}(x)A_{\nu}(y)}[g,A] = \sqrt{-g(x)}g^{\mu\nu}\delta(x-y). \tag{3-86}$$

由此可写出Landau-DeWitt规范固定项：

$$\mathcal{L}_{\text{GF}} = \frac{1}{2\alpha}F^2_{\lambda}[h_{\mu\nu}, a_{\mu}] - \frac{1}{2\beta}F^2[a_{\mu}]. \tag{3-87}$$

其中，规范固定条件F分为两部分，对应于广义坐标规范自由度与$U(1)$规范自由度，它们分别由下两式给出：

$$
\begin{aligned}
F_\lambda[h_{\mu\nu}, a_\mu] =& -\frac{1}{2}\int \mathrm{d}^d y \mathrm{d}^d z \Big[K_{\lambda x}^{g_{\mu\nu}(y)} G_{g_{\mu\nu}(y) g_{\rho\sigma}(z)} \kappa h_{\rho\sigma}(z) \\
&+ K_{\lambda x}^{A_\mu(y)} G_{A_\mu(y) A_\nu(z)} a_\nu(z)\Big], \qquad (3\text{-}88)\\
F[a_\mu] =& -\int \mathrm{d}^d y \mathrm{d}^d z \Big[K_{x}^{g_{\mu\nu}(y)} G_{g_{\mu\nu}(y) g_{\rho\sigma}(z)} \kappa h_{\rho\sigma}(z) \\
&+ K_{x}^{A_\mu(y)} G_{A_\mu(y) A_\nu(z)} a_\nu(z)\Big]. \qquad (3\text{-}89)
\end{aligned}
$$

这里，我们重新标度化$F_\lambda[h_{\mu\nu}, a_\mu]$与$F[a_\mu]$，以使规范条件的纯引力部分与习见的谐和规范条件相同，使纯规范场部分与习见的Lorentz规范条件相同。

由以上讨论，可以求出：

$$
\begin{aligned}
F_\lambda[h_{\mu\nu}, a_\mu] =& (\partial^\mu h_{\mu\lambda} + \tfrac{c}{2}\partial_\lambda h) - \tfrac{1}{2}\kappa(\partial_\mu \bar{A}_\lambda - \partial_\lambda \bar{A}_\mu) a^\mu, \qquad (3\text{-}90)\\
F[a_\mu] =& \partial_\mu a^\mu. \qquad (3\text{-}91)
\end{aligned}
$$

因此，规范固定项为

$$
\begin{aligned}
\mathcal{L}_{\mathrm{GF}} =& \tfrac{1}{2\alpha}(\partial^\mu h_{\mu\lambda} + \tfrac{c}{2}\partial_\lambda h)^2 + \tfrac{\kappa^2}{8\alpha}[a^\mu(\partial_\mu \bar{A}_\lambda - \partial_\lambda \bar{A}_\mu) + \bar{A}_\lambda \partial_\mu a^\mu]^2 \\
&- \tfrac{\kappa}{2\alpha}(\partial_\mu h^{\mu\lambda} + \tfrac{c}{2}\partial^\lambda h)[a^\nu(\partial_\nu \bar{A}_\lambda - \partial_\lambda \bar{A}_\nu) + \bar{A}_\lambda \partial_\nu a^\nu] \\
&- \tfrac{1}{2\beta}(\partial_\mu a^\mu)^2. \qquad (3\text{-}92)
\end{aligned}
$$

同样，我们需要考虑鬼项。它由下式给出：

$$
\begin{aligned}
\mathcal{L}_{\mathrm{ghost}} =& \int \mathrm{d}^d x \mathrm{d}^d y \Big[\bar{c}^\alpha(x) \frac{\delta F_\alpha[h_{\mu\nu}, a_\mu, x]}{\delta \epsilon^\beta(y)} c^\beta(y) + \bar{c}(x) \frac{\delta F[a_\mu, x]}{\delta \epsilon^\beta(y)} c^\beta(y) \\
&+ \bar{c}^\alpha(x) \frac{\delta F_\alpha[h_{\mu\nu}, a_\mu, x]}{\delta \epsilon(y)} c(y) + \bar{c}(x) \frac{\delta F[a_\mu, x]}{\delta \epsilon(y)} c(y)\Big]. \qquad (3\text{-}93)
\end{aligned}
$$

于是可求得：

$$
\begin{aligned}
\mathcal{L}_{\mathrm{ghost}} =& \bar{c}^\alpha[\eta_{\alpha\beta}\partial^2 - (1+c)\partial_\alpha \partial_\beta] c^\beta - \bar{c}\partial^2 c \\
&- \kappa[\bar{c}(\partial_\mu \bar{A}_\nu + \partial_\nu \bar{A}_\mu)\partial^\mu c^\nu + \bar{c}\bar{A}_\nu \partial^2 c^\nu + \bar{c}\partial_\mu \partial_\nu \bar{A}^\mu c^\nu + \tfrac{1}{2}\bar{c}^\mu \bar{F}_{\mu\nu}\partial^\nu c] \\
&- \tfrac{1}{2}\kappa^2 \bar{c}^\mu \bar{F}_{\mu\lambda}(\bar{A}_\nu \partial^\lambda c^\nu + \partial_\nu \bar{A}^\lambda c^\nu). \qquad (3\text{-}94)
\end{aligned}
$$

3.3.3 联络项

根据标量场计算的经验，为求得联络项，我们需要知道场空间Christoffel联络的非零分量：

$$\begin{aligned}
\Gamma^{g_{\lambda\kappa}(z)}_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} =& [\tfrac{1}{4}(\eta^{\mu\nu}\delta^{\rho}_{(\lambda}\delta^{\sigma}_{\kappa)} + \eta^{\rho\sigma}\delta^{\mu}_{(\lambda}\delta^{\nu}_{\kappa)}) - \delta^{(\mu}_{(\lambda}\eta^{\nu)(\rho}\delta^{\sigma)}_{\kappa)} \\
& - \tfrac{1}{4(d-2)}\eta_{\lambda\kappa}\eta^{\mu\nu}\eta^{\rho\sigma} + \tfrac{1}{2(d-2)}\eta_{\lambda\kappa}\eta^{\mu(\rho}\eta^{\sigma)\nu}] \\
& \times \delta(z-x)\delta(z-y),
\end{aligned} \tag{3-95}$$

$$\Gamma^{A_\lambda(z)}_{g_{\mu\nu}(x)A_\rho(y)} = \tfrac{1}{4}[g^{\mu\nu}(z)\delta^{\rho}_{\lambda} - g^{\rho\mu}(z)\delta^{\nu}_{\alpha} - g^{\rho\nu}(z)\delta^{\mu}_{\lambda}]\delta(z-x)\delta(z-y), \tag{3-96}$$

$$\Gamma^{g_{\lambda\kappa}(z)}_{A_\mu(x)A_\nu(y)} = \tfrac{\kappa^2}{2}[\delta^{(\mu}_{\lambda}\delta^{\nu)}_{\kappa} - \tfrac{1+c}{2+dc}g_{\lambda\kappa}(x)g^{\mu\nu}(x)]\delta(z-x)\delta(z-y). \tag{3-97}$$

以及作用量对各场量一阶泛函微商在背景场处的取值：

$$\frac{\delta S_{\text{Y-M}}}{\delta(\kappa h_{\lambda\kappa})} = -\tfrac{1}{4}[\tfrac{1}{2}\eta^{\lambda\kappa}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta} - 2\bar{F}^{\lambda\beta}\bar{F}^{\kappa}{}_{\beta}], \tag{3-98}$$

$$\frac{\delta S_h}{\delta(\kappa h_{\lambda\kappa})} = 0, \tag{3-99}$$

$$\frac{\delta S_\Lambda}{\delta(\kappa h_{\lambda\kappa})} = -\frac{\Lambda}{\kappa^2}\eta^{\lambda\kappa}, \tag{3-100}$$

$$\frac{\delta S_{\text{Y-M}}}{\delta a_\mu} = (\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\bar{A}_\nu. \tag{3-101}$$

由此，可以计算出：

$$\begin{aligned}
& -\tfrac{1}{2}\Gamma^{g}_{gg}(\mathcal{L}_{\text{Y-M}} + \mathcal{L}_h + \mathcal{L}_\Lambda)_{,(\kappa h)}(\kappa h)^2 \\
=& \tfrac{1}{2}\kappa^2 h_{\mu\nu}h_{\rho\sigma}[\tfrac{1}{4(d-2)}(\tfrac{d}{8}\bar{F}^2 + \tfrac{d-4}{\kappa^2}\Lambda)(\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu}) \\
& - \tfrac{1}{2}(\tfrac{1}{4}(\eta^{\mu\nu}\bar{F}^{\rho\lambda}\bar{F}^{\sigma}{}_{\lambda} + \eta^{\rho\sigma}\bar{F}^{\mu\lambda}\bar{F}^{\nu}{}_{\lambda}) + \bar{F}^{\lambda(\mu}\eta^{\nu)(\rho}\bar{F}^{\sigma)}{}_{\lambda})] \\
=& \tfrac{d-4}{8(d-2)}\Lambda(h^2 - 2h^{\mu\nu}h_{\mu\nu}) + \tfrac{\kappa^2}{8}[\tfrac{d}{8(d-2)}\bar{F}^2(h^2 - 2h^{\mu\nu}h_{\mu\nu}) \\
& - hh_{\mu\nu}\bar{F}^{\mu\lambda}\bar{F}^{\nu}{}_{\lambda} - 2h^{\rho}_{\mu}h_{\rho\nu}\bar{F}^{\lambda\mu}\bar{F}^{\nu}{}_{\lambda}],
\end{aligned} \tag{3-102}$$

$$-\Gamma^{A}_{gA}(\mathcal{L}_{\text{Y-M}})_{,a}(\kappa h)a = -\tfrac{\kappa}{4}(ha_\lambda - 2h^{\rho}_{\lambda}a_\rho)\partial_\kappa\bar{F}^{\kappa\lambda}, \tag{3-103}$$

$$-\tfrac{1}{2}\Gamma^{g}_{AA}(\mathcal{L}_{\text{Y-M}} + \mathcal{L}_h + \mathcal{L}_\Lambda)_{,(\kappa h)}a^2 = \tfrac{\kappa^2}{16}[\tfrac{1}{2}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta}a^2 - 2\bar{F}^{\mu\beta}\bar{F}^{\nu}{}_{\beta}a_\mu a_\nu] + \tfrac{1}{4}\Lambda a^2. \tag{3-104}$$

3.3.4 相关Feynman规则

我们的任务是计算耦合常数e的一圈图修正，因此只需计算规范场两点顶角函数的一圈图即可。这意味着，我们只需知道Lagrangian中这样的项即可：1)关于背景规范场的次数不高于二次；2)关于量子涨落场的幂次为二次。注意到这一点可以避免很多无关的Feynman规则的计算，从而显著减少计算量。

我们现在从已经得到的Lagrangian中取出满足以上要求的项。为此，将完整的规范场A_μ分出它的背景场$\bar{A}_\mu$ 与量子涨落场a_μ：

$$\bar{A}_\mu = A_\mu + a_\mu, \tag{3-105}$$

从而有

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{3-106}$$

其中，我们定义了背景场强张量$\bar{F}_{\mu\nu}$为：

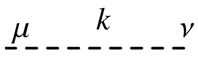
$$\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu. \tag{3-107}$$

于是，由(3-78)式，可得：

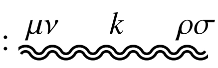
$$\begin{aligned}\mathcal{L}_{\mathrm{cQED}} =& -\tfrac{1}{4}\big[\bar{F}^2 + (\partial_\mu a_\nu - \partial_\nu a_\mu)^2\big] \\ & - \tfrac{1}{4}\kappa\big[h\bar{F}_{\mu\nu}(\partial^\mu a^\nu - \partial^\nu a^\mu) - 4h^{\mu\lambda}\bar{F}_{\mu\nu}(\partial_\rho a^\nu - \partial^\nu a_\rho)\big] \\ & - \tfrac{1}{4}\kappa^2\big[\tfrac{1}{8}(h^2 - 2h_{\rho\sigma}h^{\rho\sigma})\bar{F}^2 + (2h^\mu_\alpha h^{\alpha\rho} - hh^{\mu\rho})\bar{F}_{\mu\nu}\bar{F}_\rho{}^\nu \\ & \quad + h^{\mu\rho}h^{\nu\sigma}\bar{F}_{\mu\nu}\bar{F}_{\rho\sigma}\big] + \text{irrelevant terms}.\end{aligned} \tag{3-108}$$

由以上各Lagrangian，可以读出相关的Feynman规则如下：

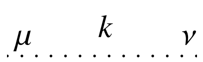
传播子 **这**里我们取真空常数$\Lambda = 0$以化简计算，因为Λ对我们所关心的二次发散无贡献。

$\varphi :$ 

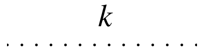
$$= \frac{-i}{k^2}\Big[\eta_{\mu\nu} - (1-\beta)\frac{k_\mu k_\nu}{k^2}\Big]$$

$h_{\mu\nu} :$ 

$$= \frac{i}{k^2}\Big[(2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}) - (1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)}}{k^2}\Big].$$

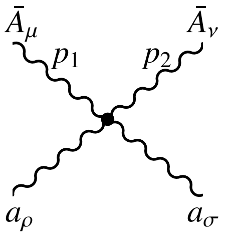
$c_\mu :$ 

$$= \frac{i}{k^2}\eta_{\mu\nu}$$

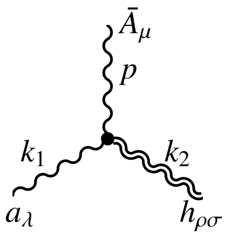
$c:$ 

$= \frac{i}{k^2}$

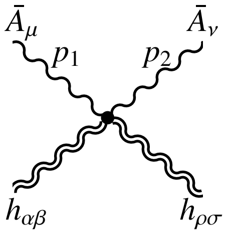
顶角



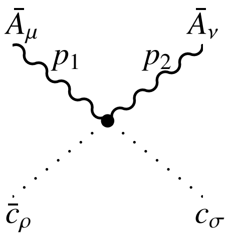
$$
\begin{aligned}
=&\frac{i\kappa^2}{4\alpha}[2p_1^{(\rho}p_2^{\sigma)}\eta^{\mu\nu} + 2p_1\cdot p_2\eta^{\mu(\rho}\eta^{\sigma)\nu} - p_1^\nu p_2^\sigma \eta^{\mu\rho}\\
&\quad - p_1^\rho p_2^\mu \eta^{\nu\sigma} - p_1^\nu p_2^\rho\eta^{\mu\sigma} - p_1^\sigma p_2^\mu \eta^{\nu\rho}]\\
&- \frac{i\kappa^2}{4}[-2(p_1^{(\rho}p_2^{\sigma)}\eta^{\mu\nu} + p_1\cdot p_2 \eta^{\mu(\rho}\eta^{\sigma)\nu} - p_1^{(\rho}\eta^{\sigma)\nu}p_2^\mu - p_2^{(\rho}\eta^{\sigma)\mu}p_1^\nu)\\
&\quad + (p_1\cdot p_2\eta^{\mu\nu} - p_1^\nu p_2^\mu)\eta^{\rho\sigma}]
\end{aligned}
$$



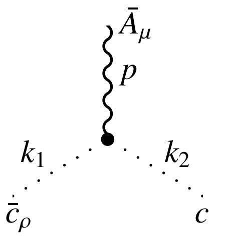
$$
\begin{aligned}
=&\frac{i\kappa}{4}[\eta^{\rho\sigma}(p_\alpha\delta^\mu_\beta - p_\beta\delta^\mu_\alpha)(k_2^\alpha\eta^{\beta\lambda} - k_2^\beta\eta^{\alpha\lambda})\\
&- 4\eta^{\alpha(\rho}\eta^{\sigma)\gamma}(p_\alpha\delta^\mu_\beta - p_\beta\delta^\mu_\alpha)(k_{2\gamma}\eta^{\beta\lambda} - k_2^\beta\delta^\lambda_\gamma)]\\
&- \frac{i\kappa}{2\alpha}(k_1^{(\rho}\eta^{\sigma)\gamma} - \frac{1}{2} k_1^\gamma\eta^{\rho\sigma})(p^\lambda\delta^\mu_\gamma - p_\gamma\eta^{\mu\lambda})
\end{aligned}
$$



$$
\begin{aligned}
=&\frac{i\kappa^2}{8}\frac{d-4}{d-2}(\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu})(p_1\cdot p_2\eta^{\alpha\beta} - p_1^\beta p_2^\alpha)\\
+&\frac{i\kappa^2}{8}[2\eta^{\lambda(\rho}\eta^{\sigma)(\mu}\eta^{\nu)\xi} + 2\eta^{\lambda(\mu}\eta^{\nu)(\rho}\eta^{\sigma)\xi} - \eta^{\rho\sigma}\eta^{\lambda(\mu}\eta^{\nu)\xi} - \eta^{\mu\nu}\eta^{\lambda(\rho}\eta^{\sigma)\xi}]\\
&\times[(p_{1\lambda}\delta^\alpha_\kappa - p_{1\kappa}\delta^\alpha_\lambda)(p_2^\kappa\eta^{\kappa\beta} - p_2^\kappa\delta^\beta_\xi) - (p_{2\lambda}\delta^\beta_\kappa - p_{2\kappa}\delta^\beta_\lambda)(p_{1\xi}\eta^{\kappa\alpha} - p_1^\kappa\delta^\alpha_\xi)]\\
&\frac{i\kappa^2}{4}[\eta^{\lambda(\rho}\eta^{\sigma)\xi}h^{\kappa\eta} + h^{\lambda\xi}\eta^{\kappa(\rho}\eta^{\sigma)\eta}]\\
&\times[(p_{1\lambda}\delta^\alpha_\kappa - p_{1\kappa}\delta^\alpha_\lambda)(p_{2\xi}\delta^\beta_\eta - p_{2\eta}\delta^\beta_\xi) + (p_{2\lambda}\delta^\beta_\kappa - p_{2\kappa}\delta^\beta_\lambda)(p_{1\xi}\delta^\alpha_\eta - p_{1\eta}\delta^\alpha_\xi)]
\end{aligned}
$$



$$= -\frac{i\kappa^2}{2}[p_1^\rho k^\nu\eta^{\mu\rho} + p_1^\sigma p_2^\rho\eta^{\mu\nu} - p_2\cdot k_2\eta^{\mu\sigma}\eta^{\nu\rho} - p_1^\sigma p_2^\mu\eta^{\rho\nu}] + (p_1\mu \leftrightarrow p_2\nu)$$



$$= i\kappa(p^\rho k_2^\mu - p\cdot k_2\eta^{\mu\rho})$$

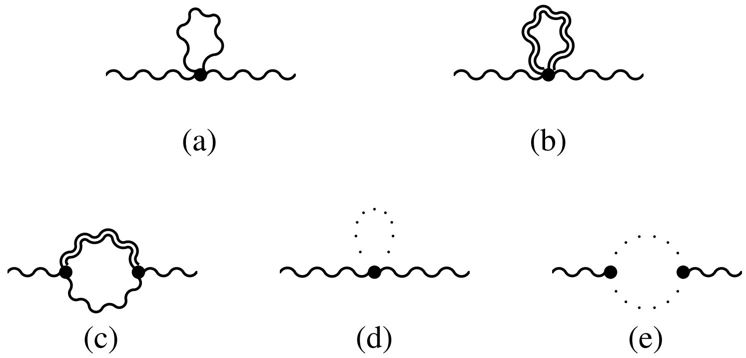
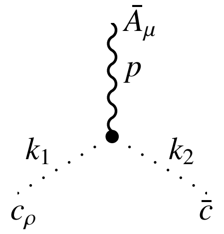


图 3.3　光子自能一圈图



$$= \frac{\mathrm{i}}{2}\kappa[(p\cdot k_1 + k_1^2)\eta^{\mu\rho} + p^m k_1^a + p^m p^a]$$

3.3.5　光子自能

如前所述，我们需要计算的量即两点顶角函数的一圈图修正，如图3.3所示。

使用截断正规化与最小剪除，并使用Tracer程序化简计算[22],可得各图的领头项为：

$$(a) = \frac{(3+\beta)\mathrm{i}}{128\alpha\pi^2}(p_\mu p_\nu - p^2 g_{\mu\nu})\kappa^2\mu^2; \tag{3-109}$$

$$(b) = \frac{3(1+\alpha)\mathrm{i}}{32\pi^2}(p_\mu p_\nu - p^2 g_{\mu\nu})\kappa^2\mu^2; \tag{3-110}$$

$$(c) = -\frac{\mathrm{i}}{16\pi^2}[\frac{3\alpha}{2} + \frac{3+\beta}{8\alpha}](p_\mu p_\nu - p^2 g_{\mu\nu})\kappa^2\mu^2; \tag{3-111}$$

$$(d) = -\frac{\mathrm{i}}{16\pi^2}(p_\mu p_\nu - p^2 g_{\mu\nu})\kappa^2\mu^2; \tag{3-112}$$

$$(e) = -\frac{\mathrm{i}}{64\pi^2}(p_\mu p_\nu - p^2 g_{\mu\nu})\kappa^2\mu^2. \tag{3-113}$$

对其求和，可得：

$$\Gamma^{(2)}_{\mu\nu}(p) = \frac{\mathrm{i}}{64\pi^2}(p_\mu p_\nu - p^2\eta_{\mu\nu})\kappa^2\mu^2. \tag{3-114}$$

于是可得β函数的领头项为：

$$\beta(e) = -\frac{\kappa^2 e\mu^2}{64\pi^2}. \tag{3-115}$$

3.3.6 Einstein-Yang-Mills理论

不难将以上分析推广到Einstein-Yang-Mills理论的情形。为此，我们只需分析以上各图在非Abel规范场时所获得的修改即可。具体分述如下：

(a) 引力子圈不变，因为该图只涉及引力作用。

(b) 光子圈图来自两部分的贡献。其一是规范固定，观察对应的顶角，可知其结构形如（为突出规范指标，我们在此省略Lorentz指标）：

$$F^a F^b a_a a_b$$

因此在外线指标给定的情形下，内圈指标亦确定。从而结果与Einstein-Maxwell理论相同。另一部分贡献来自联络项，其结构为：

$$F^a F^a a_b a_b$$

可见该部分对内圈指标求和。因此将该部分贡献推广到Einstein-Yang-Mills理论时，需乘以规范群伴随表示的维数。对于$SU(N)$群，即N^2-1。但从圈图结果不难看出，联络项在此贡献为零。因此，总体上说，光子圈图(b)不变。

(c) 引力子-光子圈不变。其原因是，引力子不带规范荷，故内圈规范场的规范指标与外线相同。

(d) 引力鬼圈图不变。因为该图只涉及引力作用。

(e) 引力鬼-光子鬼圈不变。原因与(c)相似：引力鬼不带规范荷。

综上分析，引力对Yang-Mills理论β函数的一圈图修正在领头阶与其对QED的修正完全相同。设此时的规范相互作用耦合常数为g，则我们可以立即写出β函数一圈图水平的领头阶为：

$$\beta(g) = -\frac{\kappa^2 g\mu^2}{64\pi^2}. \tag{3-116}$$

第 4 章　结论与展望

在第三章中，我们获得了在一圈图近似下，量子引力效应对所有规范理论β函数的最高阶修正。即：

$$\beta(g) = -\frac{\kappa^2 g\mu^2}{64\pi^2}. \tag{4-1}$$

该修正有如下几个特点：1)幂次跑动。由于负量纲参量κ的存在，β函数中出现了正比于能量尺度μ平方的项。这意味着，在能量低于、但接近Planck尺度之时，所有规范耦合常数将快速趋于零。2)与规范群结构无关。即，无论是Abel规范场还是非Abel规范场，量子引力带来的修正都相同，这一点至少在最高阶的一圈图近似下成立。这与不考虑引力的情形有定性的区别。如图4.1与图4.2所示。

从图4.2中可见，强、弱与电磁三种基本相互作用的规范耦合常数随能量尺度的跑动都表现出渐进自由，并且都在Planck能标附近趋于零。这暗示我们，当引力效应无可避免地发生作用时，三种基本相互作用即可在 Planck尺度实现大统一。我们指出，此时的统一无需借助超对称的条件。

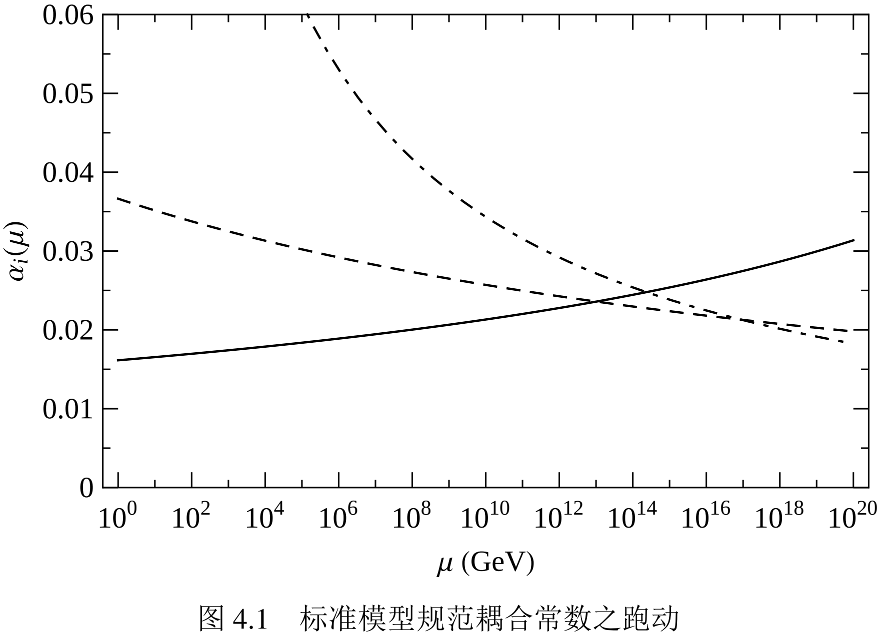


图 4.1　标准模型规范耦合常数之跑动
实线：QED；虚线：弱作用；点虚线：强作用

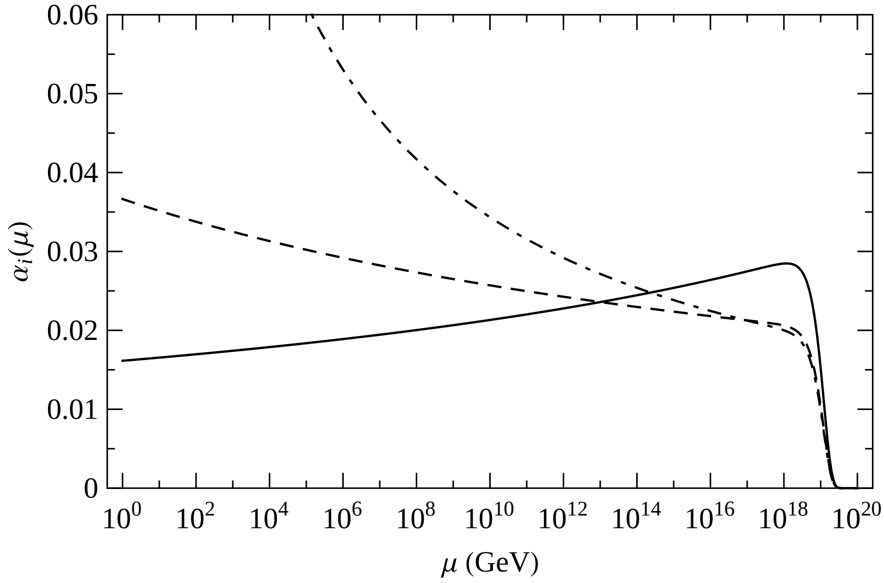


图 4.2　量子引力修正后规范耦合常数之跑动
实线：QED；虚线：弱作用；点虚线：强作用

本文计算了量子引力效应对标准模型规范耦合常数随能量跑动性质之修正的领头项。通过更为仔细地分析，尚可计算次级项，即对数发散项。对次级项的计算为判断固定点的存在与否提供了线索。另外，本文对量子引力部分的计算是在Landau-DeWitt规范下进行的。同样，我们也可以考虑在一般规范下做此计算。当然，其代价是必须要求得场空间之联络的完整形式。

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声　明

本人郑重声明：所呈交的学位论文，是本人在导师指导下，独立进行研究工作所取得的成果。尽我所知，除文中已经注明引用的内容外，本学位论文的研究成果不包含任何他人享有著作权的内容。对本论文所涉及的研究工作做出贡献的其他个人和集体，均已在文中以明确方式标明。

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附录 A 部分正文中略去的详细推导

A.1 Perturbative Expansion of the Hilbert-Einstein Action

In this section, we will perform the detailed calculations of perturbative expansion of the action for gravity in terms of a small parameter κ. At the end of our derivations, we will get a linearized theory of a massless spin-2 particle, with general covariance as expected.

In general relativity, the action for pure gravity is given by

$$S = S_G + S_\Lambda, \tag{A-1}$$

where S_G is the conventional Hilbert action, and S_Λ is the cosmological term:

$$S_G = \frac{1}{16\pi G}\int \mathrm{d}^4x\ \sqrt{-g}R = \frac{1}{\kappa^2}\int \mathrm{d}^4x\ \sqrt{-g}R, \tag{A-2}$$

$$S_\Lambda = -\frac{2}{\kappa^2}\int \mathrm{d}^4x\ \sqrt{-g}\Lambda. \tag{A-3}$$

In which G is the Newton constant, $\kappa = \sqrt{16\pi G}$ is the (typically small) parameter in which we will expand the action, and Λ is the cosmological constant.

The form of the action will be quite different when different conventions of various quantities are taken. Here we adopt the following set of conventions:

$$\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} \tag{A-4}$$

$$R = g^{\mu\nu}R_{\mu\nu}; \tag{A-5}$$

$$R_{\kappa\nu} = R^\lambda{}_{\kappa\lambda\nu}; \tag{A-6}$$

$$R^\lambda{}_{\kappa\mu\nu} = \Gamma^\lambda_{\kappa\mu,\nu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\kappa\mu} - (\mu \leftrightarrow \nu); \tag{A-7}$$

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}g^{\lambda\alpha}(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha}). \tag{A-8}$$

Where a comma denote the partial derivative:

$$A_{,\mu} = \partial_\mu A = \frac{\partial}{\partial x^\mu}A; \quad A^{,\mu} = \partial^\mu A = \frac{\partial}{\partial x_\mu}A,\ \text{ etc.}$$

We take the signature of the metric to be $(+,-,-,-)$. Different conventions are also used in literature.

We will expand the Hilbert action around the flat background metric $\eta_{\mu\nu}$. Thus we split a perturbation $h_{\mu\nu}$ out of the whole metric $g_{\mu\nu}$, as follows:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{A-9}$$

Note that κ has the dimension $[\text{mass}]^{-1}$, thus $h_{\mu\nu}$ is a field of dimension 1.

The inverse of the metric, $g^{\mu\nu}$, can be expanded as

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}_{\alpha} h^{\alpha\nu} + O(\kappa^3). \tag{A-10}$$

We note that the indices of $h_{\mu\nu}$ are raised or lowered by the background metric $\eta_{\mu\nu}$. Furthermore, we will also use the notation

$$h \equiv h^{\mu}_{\mu} = \eta^{\mu\nu} h_{\mu\nu}. \tag{A-11}$$

Next, we will expand the action (A-1) step by step. The calculations are not tricky but lengthy. So before going on, let us outline the main steps. We expand the connection coefficients $\Gamma^{\mu}_{\mu\nu}$, the Ricci tensor $R_{\kappa\nu}$, the scalar curvature R, and the determinant factor $\sqrt{-g}$ in turn. We will expand each quantity in powers of κ, up to $O(\kappa^2)$.

First, expand the connection coefficients.

$$\begin{aligned}
\Gamma^{\lambda}_{\mu\nu} =& \tfrac{1}{2}\kappa(\eta^{\lambda\alpha} - \kappa h^{\lambda\alpha})(h_{\alpha\nu,\mu} + h_{\alpha\mu,\nu} - h_{\mu\nu,\alpha}) + O(\kappa^3) \\
=& \tfrac{1}{2}\kappa(h^{\lambda}_{\nu,\mu} + h^{\lambda}_{\mu,\nu} - h_{\mu\nu}{}^{,\lambda}) - \tfrac{1}{2}\kappa^2 h^{\lambda\alpha}(h_{\alpha\nu,\mu} + h_{\alpha\mu,\nu} - h_{\mu\nu,\alpha}) + O(\kappa^3) \\
=& \Gamma^{\lambda(1)}_{\mu\nu} + \Gamma^{\lambda(2)}_{\mu\nu} + O(\kappa^3).
\end{aligned} \tag{A-12}$$

Where we use $\Gamma^{\lambda(1)}_{\mu\nu}$ to denote the terms linear in κ, and $\Gamma^{\lambda(2)}_{\mu\nu}$ to denote the quadratic terms in κ, etc. That is,

$$\Gamma^{\lambda(1)}_{\mu\nu} = \tfrac{1}{2}\kappa(h^{\lambda}_{\nu,\mu} + h^{\lambda}_{\mu,\nu} - h_{\mu\nu}{}^{,\lambda}), \tag{A-13}$$

$$\Gamma^{\lambda(2)}_{\mu\nu} = -\tfrac{1}{2}\kappa^2 h^{\lambda\alpha}(h_{\alpha\nu,\mu} + h_{\alpha\mu,\nu} - h_{\mu\nu,\alpha}). \tag{A-14}$$

Similar notations will also be used in the following.

Then, we expand the Ricci tensor in powers of κ:

$$
\begin{aligned}
R_{\kappa\nu} =& \Gamma^{\lambda}_{\kappa\lambda,\nu} - \Gamma^{\lambda}_{\kappa\nu,\lambda} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\kappa\lambda} - \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\kappa\nu} \\
=& R^{(1)}_{\kappa\nu} + R^{(2)}_{\kappa\nu} + O(\kappa^3).
\end{aligned} \tag{A-15}
$$

There are two types of terms in $R_{\kappa\nu}$, the one in the form of $\partial\Gamma$ and the other in the form of $\Gamma\Gamma$. In the perturbative expansions, the leading order terms of $\partial\Gamma$ are linear in κ, while the leading order terms in $\Gamma\Gamma$ are quadratic in κ. Hence, only $\partial\Gamma$ terms contribute to $R^{(1)}_{\kappa\nu}$, that is,

$$
\begin{aligned}
R^{(1)}_{\kappa\nu} =& \tfrac{1}{2}\kappa(h_{,\kappa,\nu} + h^{\lambda}_{\kappa,\lambda,\nu} - h_{\kappa\lambda,\nu}^{\;\;,\lambda} - h^{\lambda}_{\nu,\kappa,\lambda} - h^{\lambda}_{\kappa,\nu,\lambda} + h_{\kappa\nu,\lambda}^{\;\;,\lambda}) \\
=& \tfrac{1}{2}\kappa(h_{,\kappa,\nu} - h_{\kappa\lambda,\nu}^{\;\;,\lambda} - h^{\lambda}_{\nu,\kappa,\lambda} + h_{\kappa\nu,\lambda}^{\;\;,\lambda})
\end{aligned} \tag{A-16}
$$

The contributions to $R^{(2)}_{\kappa\nu}$ come separately from $\partial\Gamma$ terms and $\Gamma\Gamma$ terms. But it's not difficult to see that the contribution from $\partial\Gamma$ is a total divergence, thus we will not bother to write them out explicitly. Instead, we find the following expression is enough for the further calculations:

$$
\begin{aligned}
R^{(2)}_{\kappa\nu} =& (\text{total divergence from } \partial\Gamma) + \Gamma^{\lambda(1)}_{\nu\sigma}\Gamma^{\sigma(1)}_{\kappa\lambda} - \Gamma^{\lambda(1)}_{\lambda\sigma}\Gamma^{\sigma(1)}_{\kappa\nu} \\
=& (\text{total divergence from } \partial\Gamma) \\
&+ \tfrac{1}{4}\kappa^2(h^{\lambda}_{\sigma,\nu} + h^{\lambda}_{\nu,\sigma} - h_{\nu\sigma}^{\;\;,\lambda})(h^{\sigma}_{\lambda,\kappa} + h^{\sigma}_{\kappa,\lambda} - h_{\kappa\lambda}^{\;\;,\sigma}) \\
&- \tfrac{1}{4}\kappa^2 h_{,\sigma}(h^{\sigma}_{\nu,\kappa} + h^{\sigma}_{\kappa,\nu} - h_{\kappa\nu}^{\;\;,\sigma}).
\end{aligned} \tag{A-17}
$$

Now, we can expand the scalar curvature R, in the following form:

$$
\begin{aligned}
R =& g^{\kappa\nu}R_{\kappa\nu} = (\eta^{\kappa\nu} - \kappa h^{\kappa\nu})R_{\kappa\nu} + O(\kappa^3) \\
=& R^{(1)} + R^{(2)} + O(\kappa^3).
\end{aligned} \tag{A-18}
$$

More explicitly, we have

$$
\begin{aligned}
R^{(1)} =& \eta^{\kappa\nu}R^{(1)}_{\kappa\nu} \\
=& \tfrac{1}{2}\kappa\eta^{\kappa\nu}(h^{\lambda}_{\lambda,\kappa,\nu} + h^{\lambda}_{\kappa,\lambda,\nu} - h_{\kappa\lambda,\nu}^{\;\;,\lambda} - h^{\lambda}_{\nu,\kappa,\lambda} - h^{\lambda}_{\kappa,\nu,\lambda} + h_{\kappa\nu,\lambda}^{\;\;,\lambda}) \\
=& \kappa(\partial^2 h - \partial^{\mu}\partial^{\nu}h_{\mu\nu}),
\end{aligned} \tag{A-19}
$$

and

$$
R^{(2)} = \eta^{\kappa\nu}R^{(2)}_{\kappa\nu} - \kappa h^{\kappa\nu}R^{(1)}_{\kappa\nu}, \tag{A-20}
$$

$$\eta^{\kappa\nu}R^{(2)}_{\kappa\nu} = \tfrac{1}{4}\kappa^2(h^{\lambda}_{\sigma,\nu} + h^{\lambda}_{\nu,\sigma} - h_{\nu\sigma}{}^{;\lambda})(h^{\sigma,\nu}_{\lambda} + h^{\sigma\nu}{}_{,\lambda} - h^{\nu,\sigma}_{\lambda})$$
$$- \tfrac{1}{4}\kappa^2 h_{,\sigma}(h^{\sigma,\nu}_{\nu} + h^{\sigma\nu}{}_{,\nu} - h^{,\sigma}) + \text{(total divergence)}$$
$$= \tfrac{1}{4}\kappa^2(h^{\mu\nu}\partial^2 h_{\mu\nu} + 2\partial^{\mu}h_{\mu\lambda}\partial_{\nu}h^{\nu\lambda} + 2h\partial^{\mu}\partial^{\nu}h_{\mu\nu} - h\partial^2 h)$$
$$+ \text{(total divergence)},$$

$$-\kappa h^{\kappa\nu}R^{(1)}_{\kappa\nu} = -\tfrac{1}{2}\kappa^2 h^{\kappa\nu}(h_{,\kappa,\nu} - h_{\kappa\lambda,\nu}{}^{,\lambda} - h^{\lambda}_{\nu,\kappa,\lambda} + h_{\kappa\nu,\lambda}{}^{,\lambda})$$
$$= -\tfrac{1}{2}\kappa^2(h\partial^{\mu}\partial^{\nu}h_{\mu\nu} + 2\partial^{\mu}h_{\mu\lambda}\partial_{\nu}h^{\nu\lambda} + h^{\mu\nu}\partial^2 h_{\mu\nu})$$
$$+ \text{(total divergence)}.$$

Thus,

$$R^{(2)} = \tfrac{1}{4}\kappa^2(-h^{\mu\nu}\partial^2 h_{\mu\nu} - 2\partial^{\mu}h_{\mu\lambda}\partial_{\nu}h^{\nu\lambda} - h\partial^2 h) + \text{(total divergence)}. \tag{A-21}$$

We still have to expand $\sqrt{-g}$. To achieve this, we make use of $\log\det A = \text{tr}\,\log A$. Then,

$$\det g_{\mu\nu} = \exp(\text{tr}\,\log g_{\mu\nu}) = \exp[\text{tr}\,\log(\eta_{\mu\nu} + \kappa h_{\mu\nu})]$$
$$= \exp[\text{tr}\,\log(\delta^{\lambda}_{\mu} + \kappa h^{\lambda}_{\mu}) - \text{tr}\,\log(\eta^{\nu\lambda})]$$
$$= -\exp[1 + \kappa h - \tfrac{1}{2}\kappa^2 h_{\mu\nu}h^{\mu\nu}]$$
$$= -[1 + \kappa h + \tfrac{1}{2}\kappa^2(h^2 - h_{\mu\nu}h^{\mu\nu})] + O(\kappa^3),$$

thus,

$$\sqrt{-\det g_{\mu\nu}} = 1 + \tfrac{1}{2}\kappa h + \tfrac{1}{8}\kappa^2(h^2 - 2h_{\mu\nu}h^{\mu\nu}) + O(\kappa^3). \tag{A-22}$$

Now we are ready to expand the combination $\sqrt{-g}R$. It reads:

$$[\sqrt{-g}R]^{(1)} = R^{(1)} = \kappa(\partial^2 h - \partial^{\mu}\partial^{\nu}h_{\mu\nu}), \tag{A-23}$$
$$[\sqrt{-g}R]^{(2)} = R^{(2)} + \tfrac{1}{2}\kappa h R^{(1)}$$
$$= \tfrac{1}{4}\kappa^2[-h^{\mu\nu}\partial^2 h_{\mu\nu} - 2\partial^{\mu}h_{\mu\lambda}\partial_{\nu}h^{\nu\lambda} + h\partial^2 h - 2h\partial^{\mu}\partial^{\nu}h_{\mu\nu}]$$
$$+ \text{(total divergence)}. \tag{A-24}$$

It's time to write the perturbative expansion of the Hilbert action. If we are allowed to drop all the total divergence, and keep only the terms up to $O(\kappa^0)$, then we get:

$$S_G = \int \mathrm{d}^4 x \mathcal{L}_h,$$

with

$$\boxed{\mathcal{L}_h = \frac{1}{4}[h\partial^2 h - h^{\mu\nu}\partial^2 h_{\mu\nu} + 2h_{\mu\lambda}\partial^\mu\partial_\nu h^{\nu\lambda} - 2h\partial^\mu\partial^\nu h_{\mu\nu}] + O(\kappa).} \tag{A-25}$$

This is precisely the Lagrangian for a massless spin-2 particle, with the symmetry of general covariance.

The perturbative expansion of cosmological term can also be written out directly now, as

$$S_\Lambda = \int \mathrm{d}^4 x \mathcal{L}_\Lambda,$$

with

$$\mathcal{L}_\Lambda = -\frac{\Lambda}{\kappa}h - \frac{\Lambda}{4}(h^2 - 2h_{\mu\nu}h^{\mu\nu}) + O(\kappa). \tag{A-26}$$

We have dropped a constant term in $\mathcal{L}_\Lambda$.

A.2 Christoffel Connections in Field Space

A.2.1 Metric

In this section, we give a detailed derivation of the Christoffel symbol for the pure gravitational field. Our starting point is the following one-parameter metric in field space:

$$G_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} = \frac{1}{\kappa^2}\sqrt{-g(x)}[g^{\mu(\rho}(x)g^{\sigma)\nu}(x) + \frac{c}{2}g^{\mu\nu}(x)g^{\rho\sigma}(x)]\delta(x-y). \tag{A-27}$$

Some conventions:

$$g(x) \equiv \det g_{\mu\nu}(x),$$

$$A_{(\mu}B_{\nu)} \equiv \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu),$$

Delta function $\delta(x)$ are defined as

$$\int \mathrm{d}^d x \delta(x-y) f(y) = f(x),$$

thus it transforms as a scalar density under general coordinates transformations.

Here are some frequently used formulae in the following derivations:

$$\frac{\delta}{\delta g^{\mu\nu}(x)} = -g_{\mu\rho}(x)g_{\nu\sigma}(x)\frac{\delta}{\delta g_{\rho\sigma}(x)};$$

$$\frac{\delta}{\delta g_{\mu\nu}(y)}g_{\rho\sigma}(x) = \delta_\rho^{(\mu}\delta_\sigma^{\nu)}\delta(x-y);$$

$$\frac{\delta}{\delta g_{\mu\nu}(y)}g^{\rho\sigma}(x) = -g^{\rho(\mu}g^{\nu)\sigma}\delta(x-y);$$

$$\frac{\delta}{\delta g_{\mu\nu}(y)}g(x) = g(x)g^{\mu\nu}(x)\delta(x-y).$$

Now we begin the derivation. The Christoffel symbol Γ^k_{ij} for metric G_{ij} is:

$$\Gamma^k_{ij} = \tfrac{1}{2}G^{kl}(G_{li,j} + G_{lj,i} - G_{ij,l}). \tag{A-28}$$

Thus we need to work out the inverse metric G^{ij}. It is not difficult to show that,

$$\begin{aligned} &G^{g_{\mu\nu}(x),g_{\rho\sigma}(y)} \\ =&\kappa^2|g(x)|^{-1/2}[g_{\mu(\rho}(x)g_{\sigma)\nu}(x) - \tfrac{c}{2+dc}g_{\mu\nu}(x)g_{\rho\sigma}(x)]\delta(x-y). \end{aligned} \tag{A-29}$$

Now we calculate the following quantity:

$$\begin{aligned} \Gamma^k_{ij} =&\Gamma^{g_{\lambda\tau}(z)}_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} \\ =&\frac{1}{2}\int \mathrm{d}^d w\, G^{g_{\lambda\tau}(z),g_{\alpha\beta}(w)}\Big[\frac{\delta}{\delta g_{\rho\sigma}(y)}G_{g_{\alpha\beta}(w),g_{\mu\nu}(x)} \\ &+ \frac{\delta}{\delta g_{\mu\nu}(x)}G_{g_{\alpha\beta}(w),g_{\rho\sigma}(y)} + \frac{\delta}{\delta g_{\alpha\beta}(w)}G_{g_{\mu\nu}(x),g_{\rho\sigma}(y)}\Big]. \end{aligned}$$

Step by step.

$$\begin{aligned} G_{li,j} =& \frac{\delta}{\delta g_{\rho\sigma}(y)}G_{g_{\alpha\beta}(w),g_{\mu\nu}(x)} \\ =&\frac{\delta}{\delta g_{\rho\sigma}(y)}\Big\{\tfrac{1}{\kappa^2}|g(w)|^{1/2}[g^{\alpha(\mu}(w)g^{\nu)\beta}(w) + \tfrac{c}{2}g^{\alpha\beta}(w)g^{\mu\nu}(w)]\Big\} \\ =&\tfrac{1}{\kappa^2}\sqrt{-g(w)}[\tfrac{1}{2}g^{\rho\sigma}g^{\alpha(\mu}g^{\nu)\beta} - g^{\alpha(\rho}g^{\sigma)(\mu}g^{\nu)\beta} - g^{\alpha(\mu}g^{\nu)(\rho}g^{\sigma)\beta} \\ &+ \tfrac{c}{2}(\tfrac{1}{2}g^{\rho\sigma}g^{\alpha\beta}g^{\mu\nu} - g^{\alpha(\rho}g^{\sigma)\beta}g^{\mu\nu} - g^{\alpha\beta}g^{\mu(\rho}g^{\sigma)\nu})] \\ &\times\delta(w-x)\delta(w-y). \end{aligned}$$

Then,

$$
\begin{aligned}
&G_{li,j} + G_{lj,i} - G_{ij,l} \\
=&\tfrac{1}{\kappa^2}|g(w)|^{1/2}[\tfrac{1}{2}(g^{\rho\sigma}g^{\alpha(\mu}g^{\nu)\beta} + g^{\mu\nu}g^{\alpha(\rho}g^{\sigma)\beta} - g^{\alpha\beta}g^{\mu(\rho}g^{\sigma)\nu}) \\
&- 2g^{\alpha(\mu}g^{\nu)(\rho}g^{\sigma)\beta} - 2g^{\alpha(\rho}g^{\sigma)(\mu}g^{\nu)\beta} + g^{\mu(\alpha}g^{\beta)(\rho}g^{\sigma)\nu} + g^{\mu(\rho}g^{\sigma)(\alpha}g^{\beta)\nu} \\
&+ \tfrac{c}{2}(\tfrac{1}{2}g^{\alpha\beta}g^{\mu\nu}g^{\rho\sigma} - 2g^{\alpha\beta}g^{\mu(\rho}g^{\sigma)\nu})] \\
&\times \delta(w-x)\delta(w-y).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Gamma^k_{ij} =&\tfrac{1}{2}G^{kl}(G_{li,j} + G_{lj,i} - G_{ij,l}) \\
=&\tfrac{1}{2}\{\tfrac{1}{2}(g^{\rho\sigma}\delta^{\mu}_{(\lambda}\delta^{\nu}_{\tau)} + g^{\mu\nu}\delta^{\rho}_{(\lambda}\delta^{\sigma}_{\tau)} - g_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu}) \\
&- 2\delta^{(\mu}_{(\lambda}g^{\nu)(\rho}\delta^{\sigma)}_{\tau)} - 2\delta^{(\rho}_{(\lambda}g^{\sigma)(\mu}\delta^{\nu)}_{\tau)} + \delta^{\mu}_{(\lambda}\delta^{(\rho}_{\tau)}g^{\sigma)\nu} + g^{\mu(\rho}\delta^{\sigma)}_{(\lambda}\delta^{\nu}_{\tau)} \\
&+ \tfrac{c}{2}(\tfrac{1}{2}g_{\lambda\tau}g^{\mu\nu}g^{\rho\sigma} - 2g_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu}) \\
&- \tfrac{c}{2+dc}[\tfrac{1}{2}(2g_{\lambda\tau}g^{\rho\sigma}g^{\mu\nu} - dg_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu}) - 2g_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu} \\
&+ \tfrac{c}{2}(\tfrac{1}{2}dg_{\lambda\tau}g^{\mu\nu}g^{\rho\sigma} - 2dg_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu})]\} \\
&\times \delta(z-x)\delta(z-y),
\end{aligned}
$$

which can be simplified to

$$
\begin{aligned}
\Gamma^{g_{\lambda\tau}(z)}_{g_{\mu\nu}(x),g_{\rho\sigma}(y)} =&[\tfrac{1}{4}(g^{\mu\nu}\delta^{\rho}_{(\lambda}\delta^{\sigma}_{\tau)} + g^{\rho\sigma}\delta^{\mu}_{(\lambda}\delta^{\nu}_{\tau)}) - \delta^{(\mu}_{(\lambda}g^{\nu)(\rho}\delta^{\sigma)}_{\tau)} \\
&- \tfrac{c}{4(2+dc)}g_{\lambda\tau}g^{\mu\nu}g^{\rho\sigma} - \tfrac{1}{2(2+dc)}g_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu}] \\
&\times \delta(z-x)\delta(z-y).
\end{aligned}
\tag{A-30}
$$

A.2.2 Scalar

Consider a scalar field. Its metric in the field space is given by

$$
G_{\phi(x)\phi(y)} = \sqrt{-g(x)}\delta(x-y). \tag{A-31}
$$

Then the potentially non-vanishing coefficients of connection are

$$
\Gamma^{\phi(z)}_{g_{\mu\nu}(x)g_{\rho\sigma}(y)}, \quad \Gamma^{g_{\lambda\kappa}(z)}_{g_{\mu\nu}(x)\phi(y)}, \quad \Gamma^{\phi(z)}_{g_{\mu\nu}(x)\phi(y)}, \quad \Gamma^{g_{\lambda\kappa}(z)}_{\phi(x)\phi(y)}.
$$

It's not difficult to see that the first two coefficients vanish. Then we only need to evaluate the remaining two.

$$\begin{aligned}
\Gamma^{\phi(z)}_{g_{\mu\nu}(x)\phi(y)} &= \tfrac{1}{2}\int \mathrm{d}^d w\, G^{\phi(z)\phi(w)} \frac{\delta}{\delta g_{\mu\nu}(x)} G_{\phi(w)\phi(y)} \\
&= \tfrac{1}{2}\int \mathrm{d}^d w\, \frac{1}{\sqrt{-g(z)}} \frac{\delta}{\delta g_{\mu\nu}(x)} \sqrt{-g(w)}\delta(w-y)\delta(z-w) \\
&= \tfrac{1}{4} g^{\mu\nu}(z)\delta(z-x)\delta(z-y). \qquad \text{(A-32)}
\end{aligned}$$

$$\begin{aligned}
\Gamma^{g_{\lambda\kappa}(z)}_{\phi(x)\phi(y)} &= -\tfrac{1}{2}\int \mathrm{d}^d w\, G^{g_{\lambda\kappa}(z) g_{\mu\nu}(w)} \frac{\delta}{\delta g_{\mu\nu}(w)} G_{\phi(x)\phi(y)} \\
&= -\tfrac{1}{2}\int \mathrm{d}^d w\, \frac{\kappa^2}{\sqrt{-g(z)}} [g_{\mu(\lambda}(z) g_{\kappa)\nu}(z) - \tfrac{c}{2+dc} g_{\mu\nu}(z) g_{\lambda\kappa}(z)] \\
&\quad \times \frac{\delta}{\delta g_{\mu\nu}(w)} \sqrt{-g(x)}\delta(x-y)\delta(z-w) \\
&= -\tfrac{\kappa^2}{2(2+dc)} g_{\lambda\kappa}(z)\delta(z-x)\delta(z-y). \qquad \text{(A-33)}
\end{aligned}$$

A.2.3 Gauge Field

Next we consider the Yang-Mills fields. The corresponding metric in the field space is

$$G_{A^a_\mu(x) A^b_\nu(y)} = \sqrt{-g(x)}\delta^{ab} g^{\mu\nu}(x)\delta(x-y). \qquad \text{(A-34)}$$

By the same arguments in last subsection, we see that the only non-vanishing coefficients of connections are

$$\Gamma^{A^c_\lambda(z)}_{g_{\mu\nu}(x) A^b_\rho(y)}, \quad \Gamma^{g_{\lambda\kappa}(z)}_{A^a_\mu(x) A^b_\nu(y)}. \qquad \text{(A-35)}$$

Thus,

$$\begin{aligned}
\Gamma^{A^c_\lambda(z)}_{g_{\mu\nu}(x) A^b_\rho(y)} &= \tfrac{1}{2}\int \mathrm{d}^d w\, G^{A^c_\lambda(z) A^d_\alpha(w)} \frac{\delta}{\delta g_{\mu\nu}(x)} G_{A^d_\alpha(w) A^b_\rho(y)} \\
&= \tfrac{1}{2}\int \mathrm{d}^d w\, \frac{1}{\sqrt{-g(z)}} \frac{\delta}{\delta g_{\mu\nu}(x)} [\sqrt{-g(w)} g^{\alpha\rho}(w)] \\
&\quad \times g_{\lambda\alpha}(z)\delta^{cd}\delta^{db}\delta(w-y)\delta(z-w) \\
&= \tfrac{1}{4}[g^{\mu\nu}(z)\delta^\rho_\lambda - g^{\rho\mu}(z)\delta^\nu_\alpha - g^{\rho\nu}(z)\delta^\mu_\lambda]\delta^{bc}\delta(z-x)\delta(z-y). \qquad \text{(A-36)}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{g_{\lambda\kappa}(z)}_{A^a_\mu(x)A^b_\nu(y)} &= -\tfrac{1}{2}\int \mathrm{d}^d w\, G^{g_{\lambda\kappa}(z)g_{\rho\sigma}(w)} \frac{\delta}{\delta g_{\mu\nu}(w)} G_{A^a_\mu(x)A^b_\nu(y)} \\
&= -\tfrac{1}{2}\int \mathrm{d}^d w\, \frac{\kappa^2}{\sqrt{-g(z)}}[g_{\rho(\lambda}(z)g_{\kappa)\sigma}(z) - \tfrac{c}{2+dc} g_{\rho\sigma}(z)g_{\lambda\kappa}(z)] \\
&\quad \times \frac{\delta}{\delta g_{\mu\nu}(w)}[\sqrt{-g(x)}\delta^{ab} g^{\mu\nu}(x)]\delta(x-y)\delta(z-w) \\
&= \tfrac{\kappa^2}{2}[\delta^{(\mu}_\lambda \delta^{\nu)}_\kappa - \tfrac{1+c}{2+dc} g_{\lambda\kappa}(x)g^{\mu\nu}(x)]\delta^{ab}\delta(z-x)\delta(z-y). \qquad \text{(A-37)}
\end{aligned}
$$

A.3 1-Loop Diagrams in Scalar Theory

A.3.1 Two-Point Vertex Function

(a) The Scalar Loop

$$
(a) = \tfrac{1}{2}[\lambda + \kappa^2(\tfrac{1}{4} - \tfrac{1}{2\alpha})p^2 + \tfrac{1}{2}\kappa^2 m^2] \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 - \frac{d}{2(d-2)}\Lambda}. \qquad \text{(A-38)}
$$

Taking $d = 4$, and using cutoff regularization, we get

$$
\begin{aligned}
(a) =& \tfrac{i}{32\pi^2}[\lambda + \kappa^2(\tfrac{1}{4} - \tfrac{1}{2\alpha})p^2 - \tfrac{1}{2}\kappa^2 m^2] \\
&\times \Big[- M^2 + (m^2 - \Lambda)\log(1 + \tfrac{M^2}{m^2-\Lambda})\Big]. \qquad \text{(A-39)}
\end{aligned}
$$

Renormalizing the result by minimal substraction with physical scale μ, then we have

$$
(a) = \tfrac{i}{32\pi^2}[\lambda + \kappa^2(\tfrac{1}{4} - \tfrac{1}{2\alpha})p^2 - \tfrac{1}{2}\kappa^2 m^2][-\mu^2 + (m^2 - \Lambda)\log\mu^2]. \qquad \text{(A-40)}
$$

(b) The Graviton Loop

$$
\begin{aligned}
(b) =& \tfrac{1}{2}\{\tfrac{i}{8(d-2)}\kappa^2[(d-4)p^2 - dm^2][\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu}] \\
&- \tfrac{i}{4}\kappa^2[\eta^{\mu\nu}p^\rho p^\sigma + \eta^{\rho\sigma}p^\mu p^\nu] + i\kappa^2 p^{(\mu}\eta^{\nu)(\rho}p^{\sigma)}\} \\
&\times \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{i}{k^2 + (2 - \frac{d-4}{d-2})\Lambda}\Big[(2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}) \\
&\quad - (1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)}}{k^2 + (2 - \frac{d-4}{d-2})\alpha\Lambda}\Big] \qquad \text{(A-41)}
\end{aligned}
$$

Contractions of indices are listed as follows:

$$
\begin{aligned}
&[\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu}][2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}] = \tfrac{d(4+2d-2d^2)}{d-2}. \\
&[\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu}]k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)} = -dk^2.
\end{aligned}
$$

$$[\eta^{\mu\nu}p^\rho p^\sigma + \eta^{\rho\sigma}p^\mu p^\nu][2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}] = -\tfrac{8}{d-2}p^2.$$
$$[\eta^{\mu\nu}p^\rho p^\sigma + \eta^{\rho\sigma}p^\mu p^\nu]k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)} = 2(p\cdot k)^2.$$
$$p^{(\mu}\eta^{\nu)(\rho}p^{\sigma)}[2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}] = -\tfrac{4}{d-2}p^2.$$
$$p^{(\mu}\eta^{\nu)(\rho}p^{\sigma)}k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)} = \tfrac{d+2}{4}(k\cdot p)^2 + \tfrac{1}{4}k^2p^2.$$

The integral can be simplified by first taking α in the denominator $\frac{1}{k^2+\left(2-\frac{d-4}{d-2}\right)\alpha\Lambda}$ to be zero, but keep the other αs explicit. Then, after all the indices being contracted, we get

$$\begin{aligned}(\mathrm{b}) = \kappa^2\{&\tfrac{(8+16\alpha)-4(3+4\alpha)d+(5+2\alpha)d^2-d^3}{8(d-2)}p^2 - \tfrac{d^2(d-1+2\alpha)}{8(d-2)}m^2\}\\ &\times\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2+(2-\frac{d-4}{d-2})\Lambda}.\end{aligned}$$

Now taking $d = 4$ and $\alpha = 0$, and working out the integral, we get

$$(\mathrm{b}) = \tfrac{i}{16\pi^2}\kappa^2(\tfrac{3}{2}p^2 - 3m^2)\Big[-M^2 - 2\Lambda\log\big(1-\tfrac{M^2}{2\Lambda}\big)\Big]. \qquad \text{(A-42)}$$

Renormalization:

$$(\mathrm{b}) = \tfrac{i}{16\pi^2}\kappa^2(\tfrac{3}{2}p^2 - 3m^2)\Big[-\mu^2 - 2\Lambda\log\mu^2\Big]. \qquad \text{(A-43)}$$

(c) The Ghost Loop

$$(\mathrm{c}) = (-1)i\kappa^2p^\mu p^\nu\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2}\eta_{\mu\nu} = \kappa^2p^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}. \qquad \text{(A-44)}$$

Taking $d = 4$, we get

$$(\mathrm{c}) = -\tfrac{i}{16\pi^2}\kappa^2p^2M^2. \qquad \text{(A-45)}$$

Renormalization

$$(\mathrm{c}) = -\tfrac{i}{16\pi^2}\kappa^2p^2\mu^2. \qquad \text{(A-46)}$$

(d) The Graviton-Scalar Loop

$$\begin{aligned}(\mathrm{d}) = \{&i\kappa(k-p)^{(\mu}p^{\nu)} - \tfrac{i}{2\alpha}\kappa p^{(\mu}k^{\nu)}\\ &- \tfrac{i}{2}\kappa[(k-p)\cdot p + \tfrac{1}{2}p^2 - \tfrac{1}{2\alpha}p\cdot k + \tfrac{1}{2}m^2]\eta^{\mu\nu}\}\\ &\times\{i\kappa(k-p)^{(\rho}p^{\sigma)} - \tfrac{i}{2\alpha}\kappa p^{(\rho}k^{\sigma)}\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{i}{2}\kappa[(k-p)\cdot p+\tfrac{1}{2}p^2-\tfrac{1}{2\alpha}p\cdot k+\tfrac{1}{2}m^2]\eta^{\rho\sigma}\}\\
&\times\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2+(2-\frac{d-4}{d-2})\Lambda}\frac{i}{(k-p)^2-m^2+\frac{d}{2(d-2)}\Lambda}\\
&\times\Big[(2\eta_{\mu(\rho}\eta_{\sigma)\nu}-\tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma})-(1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)}}{k^2+(2-\frac{d-4}{d-2})\alpha\Lambda}\Big]
\end{aligned}
\tag{A-47}
$$

Contracting all indices, and taking $d\to 4$, we get

$$
\begin{aligned}
\text{(d)}=\tfrac{\kappa^2}{4\alpha}&\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+2\Lambda}\frac{1}{k^2+2\alpha\Lambda}\frac{1}{(k-p)^2-(m^2-\Lambda)}\\
&\times\Big[k^2p^2(k^2+2\Lambda)+\alpha(-3k^2m^4-4k^4p^2+2k^2m^2p^2+5k^2p^4\\
&-8(k\cdot p)^2(m^2+p^2)-8k^2p^2\Lambda+2(k\cdot p)(m^2+3p^2)(k^2+2\Lambda))\Big]+\mathcal{O}(\alpha).
\end{aligned}
\tag{A-48}
$$

We do not write out the explicit form for terms of order $\mathcal{O}(\alpha)$ since they will vanish after taking limit $\alpha\to 0$. We also note that only terms independent of p and proportional to p^2 are relevant, since the former contribute to the mass renormalization and the latter contribute to the wave function renormalization. Terms other than ones in these forms arise from the non-renormalizability of the theory. We will call them "irrelevant terms". Then, collect the needed terms and list them by the order in α, we get

$$
\begin{aligned}
\text{(d)}=\tfrac{1}{4\alpha}\kappa^2p^2&\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2}{k^2+2\alpha\Lambda}\frac{1}{(k-p)^2-(m^2-\Lambda)}\\
+\tfrac{1}{4}\kappa^2&\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+2\Lambda}\frac{1}{k^2+2\alpha\Lambda}\frac{1}{(k-p)^2-(m^2-\Lambda)}\\
&\times[-3k^2m^4-4k^4p^2+2k^2m^2p^2-8(k\cdot p)^2m^2-8k^2p^2\Lambda\\
&+2(k\cdot p)(m^2+3p^2)(k^2+2\Lambda)]+\text{irrelevant terms}.
\end{aligned}
\tag{A-49}
$$

Thus we need to work out the two integrals in the expression above. We call them (d1) and (d2) The first one (d1) is proportional to α^{-1}, thus it needed a careful treatment:

$$
\begin{aligned}
\text{(d1)}&=\tfrac{1}{4\alpha}\kappa^2p^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2}{k^2+2\alpha\Lambda}\frac{1}{(k-p)^2-(m^2-\Lambda)}\\
&=\tfrac{1}{4\alpha}\kappa^2p^2\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{(k'+xp)^2}{(k'^2-\Delta_1)^2}\\
&=\tfrac{1}{4\alpha}\kappa^2p^2\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{k'^2}{(k'^2-\Delta_1)^2}+\text{ir.}
\end{aligned}
\tag{A-50}
$$

with

$$k' = k - xp, \tag{A-51}$$

$$\Delta_1 = x(x-1)p^2 + xm^2 - [x + 2\alpha(1-x)]\Lambda. \tag{A-52}$$

(Here and following "ir." means irrelevant or finite terms) Then using cut-off regularization, we get

$$(\text{d1}) = -\frac{\mathrm{i}\kappa^2 p^2}{64\alpha\pi^2}M^2 + \frac{\mathrm{i}\kappa^2 p^2}{64\alpha\pi^2}\int_0^1 \mathrm{d}x\, 2\Delta_1 \log\left(1 + \tfrac{M^2}{\Delta_1}\right) + \text{ir.} \tag{A-53}$$

Renormalizing this by minimal substraction with physical scale μ, we get

$$\begin{aligned}(\text{d1}) &= -\frac{\mathrm{i}\kappa^2 p^2}{64\alpha\pi^2}\mu^2 + \frac{\mathrm{i}\kappa^2 p^2}{64\alpha\pi^2}\int_0^1 \mathrm{d}x\, 2\Delta_1 \log\mu^2 + \text{ir.}\\ &= -\frac{\mathrm{i}\kappa^2 p^2}{64\alpha\pi^2}\mu^2 + \frac{\mathrm{i}\kappa^2 p^2}{64\alpha\pi^2}[m^2 - \Lambda - 2\alpha\Lambda]\log\mu^2 + \text{ir.}\\ &= \frac{\mathrm{i}\kappa^2 p^2}{64\alpha\pi^2}[-\mu^2 + (m^2-\Lambda)\log\mu^2] + \frac{\mathrm{i}\kappa^2 p^2}{32\pi^2}\Lambda\log\mu^2 + \text{ir.}\end{aligned} \tag{A-54}$$

Now we turn to (d2). Since (d2) contains no α pole, thus we are free to set all $\alpha = 0$ at first in order to simplify the calculation. Then we get

$$\begin{aligned}(\text{d2}) = \tfrac{1}{4}\kappa^2 &\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{k^2+2\Lambda}\frac{1}{(k-p)^2-(m^2-\Lambda)}\\ &\times[-3k^2m^4 - 4k^4p^2 + 2k^2m^2p^2 - 8(k\cdot p)^2m^2\\ &\quad - 8k^2p^2\Lambda + 2(k\cdot p)(m^2+3p^2)(k^2+2\Lambda)]\\ = &\tfrac{1}{4}\kappa^2\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-3m^4 - 4k^2p^2 + 2m^2p^2 - 8p^2\Lambda}{(k^2+2\Lambda)[(k-p)^2-(m^2-\Lambda)]}\\ &\tfrac{1}{4}\kappa^2\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{2(k\cdot p)(m^2+3p^2)}{(k^2+2\alpha\Lambda)[(k-p)^2-(m^2-\Lambda)]}\\ &-2\kappa^2m^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k\cdot p)^2}{k^2(k^2+2\Lambda)[(k-p)^2-(m^2-\Lambda)]}\\ = &\tfrac{1}{4}\kappa^2\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\,\frac{-4k'^2p^2 - 3m^4 + 2m^2p^2 - 8p^2\Lambda}{(k'^2-\Delta_2)^2}\\ &+\tfrac{1}{4}\kappa^2\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\,\frac{2xp^2m^2}{(k'^2-\Delta_1)^2}\\ &-\kappa^2m^2p^2\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{k'^2}{(k'^2-\Delta_3)^3},\end{aligned} \tag{A-55}$$

with

$$k' = k - xp; \tag{A-56}$$

$$\Delta_2 = x(x-1)p^2 + xm^2 + (x-2)\Lambda; \tag{A-57}$$

$$\Delta_3 = x(x-1)p^2 + xm^2 - (x+2y)\Lambda. \tag{A-58}$$

Working out this with cut-off regularization and minimal subtraction by μ, we get

$$\begin{aligned}(\mathrm{d2}) = &\frac{\mathrm{i}\kappa^2}{64\pi^2}\int_0^1 \mathrm{d}x\left[4p^2(\mu^2 - 2\Delta_2\log\mu^2) + (-3m^4 + 2m^2p^2 - 8p^2\Lambda)\log\mu^2\right]\\ &+ \frac{\mathrm{i}\kappa^2}{64\pi^2}p^2m^2\log\mu^2 - \frac{\mathrm{i}\kappa^2}{32\pi^2}p^2m^2\log\mu^2\\ = &\frac{\mathrm{i}\kappa^2}{16\pi^2}p^2\mu^2 - \frac{\mathrm{i}\kappa^2}{16\pi^2}[p^2(\tfrac{3}{4}m^2 - \Lambda) + \tfrac{3}{4}m^4]\log\mu^2.\end{aligned} \tag{A-59}$$

Now combining (d1) and (d2), we get:

$$\begin{aligned}(\mathrm{d}) = &-\frac{\mathrm{i}\kappa^2p^2}{64\alpha\pi^2}\mu^2 + \frac{\mathrm{i}\kappa^2p^2}{64\alpha\pi^2}[m^2 - \Lambda - 2\alpha\Lambda]\log\mu^2\\ &+ \frac{\mathrm{i}\kappa^2}{16\pi^2}p^2\mu^2 - \frac{\mathrm{i}\kappa^2}{16\pi^2}[p^2(\tfrac{3}{4}m^2 - \Lambda) + \tfrac{3}{4}m^4]\log\mu^2\\ = &\frac{\mathrm{i}\kappa^2}{16\pi^2}(1 - \tfrac{1}{4\alpha})p^2\mu^2\\ &+ \frac{\mathrm{i}\kappa^2}{16\pi^2}[\tfrac{1}{4\alpha}p^2(m^2 - \Lambda) - (\tfrac{3}{4}m^2 - \tfrac{1}{2}\Lambda)p^2 - \tfrac{3}{4}m^4]\log\mu^2.\end{aligned} \tag{A-60}$$

A.3.2 Four-Point Vertex Function

(a) The Scalar Loop

$$(\mathrm{a}) = \tfrac{1}{2}i\kappa^2\lambda\tfrac{d}{4(d-2)}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2 - m^2 + \frac{d}{4(d-2)}\Lambda}. \tag{A-61}$$

Setting $d = 4$, we get

$$(\mathrm{a}) = \frac{i\kappa^2\lambda}{64\pi^2}\left[M^2 - (m^2 - \tfrac{1}{2}\Lambda)\log(1 + \tfrac{M^2}{m^2 - \frac{1}{2}\Lambda})\right]. \tag{A-62}$$

(b) The Graviton Loop

$$\begin{aligned}(\mathrm{b}) = &-\tfrac{i}{2}\kappa^2\lambda\tfrac{d}{8(d-2)}[\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu(\rho}\eta^{\sigma)\nu}]\\ &\times\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2 + (2 - \frac{d-4}{d-2})\Lambda}\left[(2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma})\right.\end{aligned}$$

$$-(1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\rho}k_{\sigma)}}{k^2+(2-\frac{d-4}{d-2})\alpha\Lambda}\Big]. \tag{A-63}$$

Contracting the indices,

$$(b) = \tfrac{1}{2}\kappa^2\lambda\tfrac{d}{8(d-2)}\big[\tfrac{d(4+2d-2d^2)}{d-2}+(1-\alpha)4d\big]\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2+(2-\frac{d-4}{d-2})\Lambda}. \tag{A-64}$$

Setting $d=4$, we get

$$(b) = \frac{3i\kappa^2\lambda}{16\pi^2}\Big[M^2+2\Lambda\log\big(1-\tfrac{M^2}{2\Lambda}\big)\Big]. \tag{A-65}$$

(c) The Loop with 5-Point Interaction

$$\begin{aligned}(c) &= \big(-\tfrac{i}{4}\kappa\lambda\eta^{\mu\nu}\big)\big(-\tfrac{i}{4}\kappa m^2\eta^{\rho\sigma}\big)\\ &\times\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2+(2-\frac{d-4}{d-2})\Lambda}\frac{i}{k^2-m^2-\frac{d}{4(d-2)}\Lambda}\\ &\times\Big[(2\eta_{\mu(\alpha}\eta_{\beta)\nu}-\tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta})-(1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\alpha}k_{\beta)}}{k^2+(2-\frac{d-4}{d-2})\alpha\Lambda}\Big].\end{aligned} \tag{A-66}$$

It's convenient to set $\alpha\to 0$ at this stage. Then:

$$(c) = -\tfrac{d-1}{2(d-2)}\lambda\kappa^2m^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2+(2-\frac{d-4}{d-2})\Lambda}\frac{1}{k^2-m^2+\frac{d}{4(d-2)}\Lambda}. \tag{A-67}$$

Setting $d=4$ and working out the integral:

$$(c) = -\frac{3i\kappa^2\lambda m^2}{64\pi^2}\int_0^1\mathrm{d}x\Big[\log\big(1+\tfrac{M^2}{\Delta}\big)-\tfrac{M^2}{M^2+\Delta}\Big]. \tag{A-68}$$

(f) The Scalar-Scalar Loop

$$(f) = \tfrac{1}{2}\big[-i\lambda+i\tfrac{d}{4(d-2)}\kappa^2m^2\big]^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big[\frac{i}{k^2-m^2+\frac{d}{4(d-2)}\Lambda}\Big]^2. \tag{A-69}$$

Setting $d=4$,

$$(f) = \frac{i}{32\pi^2}\big[\lambda-\tfrac{1}{2}\kappa^2m^2\big]^2\Big[\log\big(1+\tfrac{M^2}{m^2-\frac{1}{2}\Lambda}\big)-\tfrac{M^2}{M^2+m^2-\frac{1}{2}\Lambda}\Big] \tag{A-70}$$

(g) The Graviton-Graviton Loop

$$(g) = \tfrac{1}{2}\big[-\tfrac{d}{8(d-2)}i\kappa^2m^2\big]^2(\eta^{\mu\nu}\eta^{\rho\sigma}-2\eta^{\mu(\rho}\eta^{\sigma)\nu})(\eta^{\alpha\beta}\eta^{\lambda\kappa}-2\eta^{\alpha(\lambda}\eta^{\alpha)\beta})$$

$$\times \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{i}{k^2 + (2 - \frac{d-4}{d-2})\Lambda} \frac{i}{k^2 + (2 - \frac{d-4}{d-2})\Lambda}$$
$$\times \Big[(2\eta_{\mu(\alpha}\eta_{\beta)\nu} - \tfrac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}) - (1-\alpha)\frac{4k_{(\mu}\eta_{\nu)(\alpha}k_{\beta)}}{k^2 + (2 - \frac{d-4}{d-2})\alpha\Lambda}\Big]$$
$$\times \Big[(2\eta_{\rho(\lambda}\eta_{\kappa)\sigma} - \tfrac{2}{d-2}\eta_{\rho\sigma}\eta_{\lambda\kappa}) - (1-\alpha)\frac{4k_{(\rho}\eta_{\lambda)(\kappa}k_{\sigma)}}{k^2 + (2 - \frac{d-4}{d-2})\alpha\Lambda}\Big] \tag{A-71}$$

It's convenient to set $\alpha \to 0$ at this stage. Then:

$$(\mathrm{g}) = \tfrac{1}{2}\big[\tfrac{\kappa^2 d m^2}{8(d-2)}\big]^2 (8d^2 - 8d) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[k^2 + (2 - \frac{d-4}{d-2})\Lambda]^2}. \tag{A-72}$$

Setting $d = 4$ and working out the integral:

$$(\mathrm{g}) = \frac{3i\kappa^4 m^4}{16\pi^2}\Big[\log(1 - \tfrac{M^2}{2\Lambda}) - \tfrac{M^2}{M^2 - 2\Lambda}\Big]. \tag{A-73}$$

附录 B　标准模型：从Lagrangian到Feynman规则

B.1 Quantum Chromodynamics

B.1.1 Preliminaries

The quantum chromodynamics (QCD) is a non-Abelian gauge theory with the gauge group $SU(3)$. In the classic level, the Lagrangian of QCD is given by:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \bar{\psi}_i (i\not{D} - m)\psi_i. \tag{B-1}$$

The gauge field strength $G^a_{\mu\nu}$ is given by:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_C c^{abc} G^b_\mu G^c_\nu, \tag{B-2}$$

where g_c is the gauge coupling and f^{abc} are structure constants of $SU(3)$.

The second term in the Lagrangian (B-1) are fermions term. These fermions are called quarks. There are six groups of quarks:

$$\psi^a_i = (\ u^a,\ c^a,\ t^a,\ d^a,\ s^a,\ b^a\ ). \tag{B-3}$$

Each group lies in the fundamental representation of the gauge group $SU(3)$, and each element in a group is a Dirac fermion. The covariant derivative is given by:

$$\mathrm{D}_\mu = \partial_\mu - i g_C G^a_\mu t^a, \tag{B-4}$$

where t^as are the group generators.

Once the theory get quantized, the gauge fixing is necessary. Following the standard procedure of Fadeev-Popov method, we can define the path integral of the theory in a proper way. Then the Lagrangian of the quantized theory will acquire two additional terms:

$$\Delta\mathcal{L} = \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{ghost}} = -\frac{1}{2\xi_C}(\partial^\mu G^a_\mu)^2 + \bar{c}^a_G(-\partial^\mu D^{ac}_\mu)c^c_G. \tag{B-5}$$

Now we are ready two expand all the terms in the full Lagrangian, to get the Feynman rules of QCD.

B.1.2 Pure Gauge Part

We expand the pure gauge term in the lagrangian (B-1):

$$\begin{aligned}-\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} =& -\tfrac{1}{4}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_C c^{abc}G^b_\mu G^c_\nu)^2 \\ =& -\tfrac{1}{4}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu)^2 - g_C c^{abc}G^b_\mu G^c_\nu \partial^\mu G^{a\nu} \\ & -\tfrac{1}{4}g_C^2 c^{abc}c^{ade}G^b_\mu G^c_\nu G^{d\mu}G^{e\nu},\end{aligned} \tag{B-6}$$

and add the gauge fixing term in the lagrangian (B-5)

$$-\frac{1}{2\xi_C}(\partial^\mu G^a_\mu)^2 \tag{B-7}$$

into the pure gauge term. Then we get:

$$\begin{aligned}&-\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \tfrac{1}{2\xi_C}(\partial^\mu G^a_\mu)^2 \\ =&\tfrac{1}{2}G^a_\mu[g^{\mu\nu}\partial^2 - (1-\tfrac{1}{\xi})\partial^\mu\partial^\nu]G^a_\nu - g_C c^{abc}G^b_\mu G^c_\nu \partial^\mu G^{a\nu} \\ &-\tfrac{1}{4}g_C^2 c^{abc}c^{ade}G^b_\mu G^c_\nu G^{d\mu}G^{e\nu}.\end{aligned} \tag{B-8}$$

The first term gives the propagator of the gauge boson, or gluon:

$$\begin{aligned}&\frac{1}{2}\int \mathrm{d}^4x\, G^a_\mu(x)[g^{\mu\nu}\partial^2 - (1-\tfrac{1}{\xi})\partial^\mu\partial^\nu]G^a_\nu(x) \\ =&\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}A_\mu(k)[-k^2g^{\mu\nu} + (1-\tfrac{1}{\xi_C})k^\mu k^\nu]A_\nu(-k).\end{aligned} \tag{B-9}$$

The propagator:

$$\begin{aligned}D_{\mu\nu}(k) =& -i[-k^2g^{\mu\nu} + (1-\tfrac{1}{\xi_C})k^\mu k^\nu]^{-1} \\ =& \frac{-i}{k^2+i\epsilon}\Big[g_{\mu\nu} - (1-\xi_C)\frac{k_\mu k_\nu}{k^2}\Big].\end{aligned} \tag{B-10}$$

Now we turn to the cubic term in (B-8), and work out the corresponding Feynman vertex. The derivation will be relatively easy if we realize that the Feynman vertex can be identified as the tree-level approximation of the corresponding 3-point vertex function $\Gamma^{(3)}|_{\text{tree}}$. Since the vertex functions are generated by its generating functional, namely the effective action Γ. We also know that the effective action is just the classical action S at the tree level. Thus in momentum space, we have:

$$\Gamma^{(3)}(G^a_{1\lambda}, G^b_{2\mu}, G^c_{3\nu})\Big|_{\text{tree}}$$

$$
\begin{aligned}
=&\frac{i\delta^3 S[G]}{\delta G^a_{1\lambda}\delta G^b_{2\mu}\delta G^c_{3\nu}} = i\frac{\delta^3}{\delta G^a_{1\lambda}\delta G^b_{2\mu}\delta G^c_{3\nu}}[ig_C c^{def} G^e_\alpha G^f_\beta k^\alpha G^{d\beta}] \\
=&g_C c^{abc}[g^{\nu\lambda}(k_3-k_1)^\mu + g^{\lambda\mu}(k_1-k_2)^\nu + g^{\mu\nu}(k_2-k_3)^\lambda]. \qquad \text{(B-11)}
\end{aligned}
$$

Where we have used the total-antisymmetry property of the structure constant c^{abc}.

In the same way, we can work out the $\Gamma^{(4)}$ as:

$$
\begin{aligned}
\Gamma^{(4)}(G^a_{1\mu}, G^b_{2\nu}, G^c_{3\rho}, G^d_{4\sigma}) =& \frac{i\delta^4\Gamma}{\delta G^a_{1\mu}\delta G^b_{2\nu}\delta G^c_{3\rho}\delta G^d_{4\sigma}} \\
=-ig_C^2[&c^{eab}e^{ecd}(g^{\rho\mu}g^{\sigma\nu} - g^{\mu\sigma}g^{\nu\rho}) + c^{eac}c^{edb}(g^{\mu\sigma}g^{\rho\nu} - g^{\mu\nu}g^{\rho\sigma}) \\
&+ c^{ead}e^{ebc}(g^{\mu\nu}g^{\rho\sigma} - g^{\rho\mu}g^{\sigma\nu})]. \qquad \text{(B-12)}
\end{aligned}
$$

B.1.3 Ghosts

The ghosts' Lagrangian is:

$$
\begin{aligned}
\mathcal{L}_{\text{ghost}} =& -\bar{c}^a_G\partial^\mu \mathrm{D}^{ac}_\mu c^c_G = -c^a\partial^\mu(\delta^{ac}\partial_\mu + g_C c^{abc}G^b_\mu)c^c_G \\
=& -c^a_G\partial^2 c^a_G + g_C c^{abc}c^a\partial^\mu(G^b_\mu c^c_G). \qquad \text{(B-13)}
\end{aligned}
$$

Thus the propagators of the ghosts are:

$$
D_{ab}(k) = \frac{i\delta_{ab}}{k^2 - m^2 + i\epsilon}. \qquad \text{(B-14)}
$$

The ghost-gluon interaction:

$$
\Gamma^{(3)}(G^b_\mu, \bar{c}^a_G, c^c_G) = -g_C c^{abc}k^\mu. \qquad \text{(B-15)}
$$

B.1.4 Fermion Part

The Lagrangian for the quarks are given by:

$$
\mathcal{L}_{\text{quarks}} = \bar{\psi}_i(i\not{D} - m)\psi_i = \bar{\psi}_i(i\not{\partial} - m)\psi_i + g_C G^a_\mu t^a\bar{\psi}_i\gamma^\mu\psi_i. \qquad \text{(B-16)}
$$

The first term gives the propagators of quarks:

$$
D_{ij}(k) = \frac{i}{\not{k} - m + i\epsilon}\delta_{ij}. \qquad \text{(B-17)}
$$

The second term gives the interaction vertex for a gluon and quarks:

$$
\Gamma^{(3)}(G^a_\mu, \psi_i, \bar{\psi}_j) = ig_C t^a\gamma^\mu\delta_{ij}. \qquad \text{(B-18)}
$$

B.2 The Electroweak Theory

B.2.1 Preliminaries

We first write down the Lagrangian of the electroweak theory in a compact form

$$\mathcal{L} = \mathcal{L}_{\rm G} + \mathcal{L}_{\rm H} + \mathcal{L}_{\rm F} + \mathcal{L}_{\rm FH} \,. \tag{B-19}$$

The four terms on the right hand side are respectively the pure gauge Lagrangian, the Higgs Lagrangian, the Fermion Lagrangian, and the Fermion-Higgs interaction term. Their detailed forms will be introduced in turn in the following.

According to Faddeev-Popov method, Once the theory get quantized, we must also introduce two additional terms, namely the gauge fixing term and the ghost term, into the original Lagrangian.

$$\Delta\mathcal{L} = \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm ghost}. \tag{B-20}$$

Hence, after quantization, the whole Lagrangian will be:

$$\mathcal{L}_{\rm quantized} = \mathcal{L} + \Delta\mathcal{L} = \mathcal{L}_{\rm G} + \mathcal{L}_{\rm H} + \mathcal{L}_{\rm F} + \mathcal{L}_{\rm FH} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm ghost}. \tag{B-21}$$

This is our starting point. Since it is written in an extremely compact form, thus it needs some labor to expand this Lagrangian to a form from which we can read the Feynman rules directly. This is the task of the rest of this section. Various subsections are organized as follows. (to be added)

Before going to the detailed derivation, we'd better outline the kernel idea of electroweak theory. But this is far from complete, thus only can be treated as a list of conventions on notations.

As the Lagrangian suggest, the electroweak theory is a gauge theory with Higgs mechanism. The original gauge group of the theory is $SU(2)_{\rm W} \times U(1)_{\rm Y}$. The gauge field associated with this group are:

$$W^a_\mu, \quad a = 1, 2, 3 \,, \quad \text{and} \quad B_\mu,$$

which are associated with the group $SU(2)$ and $U(1)$, respectively.

The gauge symmetry spontaneously breaks into a subgroup $U(1)$, by a scalar field

ϕ which has a nonzero vacuum expectation value:

$$\phi = \begin{pmatrix} \pi^+ \\ \frac{1}{\sqrt{2}}(h + v + i\pi^0) \end{pmatrix}, \tag{B-22}$$

$$\phi^\dagger = \begin{pmatrix} \pi^- & \frac{1}{\sqrt{2}}(h + v - i\pi^0) \end{pmatrix} \tag{B-23}$$

Thus the vacuum expectation value of ϕ is

$$\langle |\phi| \rangle = \frac{v}{\sqrt{2}}. \tag{B-24}$$

The crucial thing is that the remaining symmetry group $U(1)$ is neither the original $U(1)_\mathrm{Y}$, nor a subgroup of $SU(2)_\mathrm{W}$, but a subgroup of the mixing of $SU(2)_\mathrm{W} \times U(1)_\mathrm{Y}$. This mixing is described by a rotation on the generators (W_μ^3, B_μ):

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}. \tag{B-25}$$

Where θ_W is called the Weinberg angle. For convenience, we introduce the following notations:

$$\cos\theta_W \equiv c_W, \qquad \sin\theta_W \equiv s_W, \qquad \tan\theta_W \equiv t_W.$$

In other cases, θ_W will be expressed explicitly, for instance, we will write $\cos 2\theta_W$.

The remaining two W fields will be linearly combined into eigenstates of electric charge in the following form:

$$\begin{cases} W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-), \\ W_\mu^2 = \frac{i}{\sqrt{2}}(W_\mu^+ - W_\mu^-). \end{cases} \tag{B-26}$$

The covariant derivatives for a field quantity lying in the fundamental representation of the gauge group is:

$$\mathrm{D}_\mu = \partial_\mu - \tfrac{1}{2} i g \tau^a W_\mu^a \phi - i g' Y B_\mu. \tag{B-27}$$

Where τ^a are Pauli's matrices, we adopt the convention:

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B-28}$$

And Y is called the super charge of the field of which we take the derivative. g and g' are corresponding couplings, in terms of which the Weinberg angle can be expressed:

$$s_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \qquad c_W = \frac{g}{\sqrt{g^2 + g'^2}}. \tag{B-29}$$

On the other hand, the unit electric charge e turns out to be:

$$e = g s_W. \tag{B-30}$$

All these relations will be frequently encountered below. We will use them without further remarks.

B.2.2 Pure Gauge Sector

The pure gauge Lagrangian is:

$$\mathcal{L}_{\mathrm{G}} = -\tfrac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{B-31}$$

Where the field strength tensors are given by:

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu; \tag{B-32}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{B-33}$$

Now we expand them:

$$\begin{aligned}
-\tfrac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} &= -\tfrac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu)^2 \\
&= -\tfrac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)^2 - \tfrac{1}{2} g\epsilon^{abc} W^b_\mu W^c_\nu (\partial^\mu W^{a\nu} - \partial^\nu W^{a\mu}) \\
&\quad - \tfrac{1}{4} g^2 \epsilon^{abc} \epsilon^{ade} W^b_\mu W^c_\nu W^{d\mu} W^{e\nu} \\
&= -\tfrac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)^2 - g\epsilon^{abc} W^b_\mu W^c_\nu \partial^\mu W^{a\nu} \\
&\quad - \tfrac{1}{4} g^2 (\delta^{bd}\delta^{ce} - \delta^{be}\delta^{dc}) W^b_\mu W^c_\nu W^{d\mu} W^{e\nu} \\
&= -\tfrac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)^2 - g\epsilon^{abc} W^b_\mu W^c_\nu \partial^\mu W^{a\nu} \\
&\quad - \tfrac{1}{4} g^2 (W^b_\mu W^{b\mu} W^c_\nu W^{c\nu} - W^b_\mu W^b_\nu W^{c\mu} W^{c\nu}).
\end{aligned}$$

Then we get the quadratic term:

$$-\tfrac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)^2 - \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2$$

$$
\begin{aligned}
=& -\tfrac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) \\
& -\tfrac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2,
\end{aligned}
\tag{B-34}
$$

the cubic term:

$$
\begin{aligned}
& -g\epsilon^{abc} W^b_\mu W^c_\nu \partial^\mu W^{a\nu} \\
=& -g[(W^2_\mu W^3_\nu - W^3_\mu W^2_\nu)\partial^\mu W^{1\nu} \\
& \quad + (W^3_\mu W^1_\nu - W^1_\mu W^3_\nu)\partial^\mu W^{2\nu} + (W^1_\mu W^2_\nu - W^2_\mu W^1_\nu)\partial^\mu W^{3\nu}] \\
=& -ig(c_W Z_\nu + s_W A_\nu) \\
& \quad \times [-W^-_\mu \partial^\mu W^{+\nu} + W^-_\mu \partial^\nu W^{+\mu} + W^+_\mu \partial^\mu W^{-\nu} - W^+_\mu \partial^\nu W^{-\mu}] \\
& -ig(W^+_\mu W^-_\nu + W^-_\mu W^+_\nu)(c_W \partial^\mu Z^\nu + s_W \partial^\mu A^\nu),
\end{aligned}
\tag{B-35}
$$

and the quartic term:

$$
\begin{aligned}
& -\tfrac{1}{4}g^2[W^b_\mu W^{b\mu} W^c_\nu W^{c\nu} - W^b_\mu W^b_\nu W^{c\mu} W^{c\nu}] \\
=& -\tfrac{1}{4}g^2[(2W^+_\mu W^{-\mu} + W^3_\mu W^{3\mu})^2 \\
& \quad -(W^+_\mu W^-_\nu + W^-_\mu W^+_\nu + W^3_\mu W^3_\nu)^2] \\
=& \tfrac{1}{2}g^2[W^+_\mu W^-_\nu W^{+\mu} W^{-\nu} - W^+_\mu W^{-\mu} W^+_\nu W^{-\nu}] \\
& -g^2 W^+_\mu W^{-\mu}[c^2_W Z_\nu Z^\nu + s^2_W A_\nu A^\nu + 2c_W s_W Z_\nu A^\nu] \\
& +\tfrac{1}{2}g^2(W^+_\mu W^-_\nu + W^-_\mu W^+_\nu) \\
& \quad \times [c^2_W Z^\mu Z^\nu + s^2_W A^\mu A^\nu + 2c_W s_W Z^\mu A^\nu].
\end{aligned}
\tag{B-36}
$$

B.2.3 Self-Interactions of Scalar

The Higgs Part of the original Lagrangian (B-19) is:

$$
\mathcal{L}_\mathrm{H} = (\mathrm{D}_\mu \phi)^\dagger (\mathrm{D}^\mu \phi) - V(\phi). \tag{B-37}
$$

We first deal with the self interactions of Higgs field:

$$
\begin{aligned}
-V(\phi) =& \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 = -\lambda[\phi^\dagger \phi - \tfrac{1}{2}v^2]^2 + \text{const.} \\
=& -\lambda[\pi^+\pi^- + \tfrac{1}{2}(\pi^0)^2 + \tfrac{1}{2}(h+v)^2 - \tfrac{v^2}{2}]^2 + \text{const.} \\
=& -\lambda v^2 h^2 - \tfrac{\lambda}{4}h^4 - \lambda v h^3 - \tfrac{\lambda}{2}h^2(\pi^0)^2 \\
& -\lambda v h(\pi^0)^2 - \lambda h^2 \pi^+\pi^- - 2\lambda v h \pi^+\pi^-
\end{aligned}
$$

$$-\lambda(\pi^+\pi^-)^2 - \frac{\lambda}{4}(\pi^0)^4 - \lambda(\pi^0)^2\pi^+\pi^- + \text{const.} \tag{B-38}$$

The expressions in each step are equal up to a constant, we note that the term "const" appearing above are not all equal.

B.2.4 Higgs-Gauge Sector

Now we come to the covariant derivative term of Higgs field. Note that the supercharge of Higgs field is $\frac{1}{2}$, thus we have:

$$\mathrm{D}_\mu\phi = \partial_\mu\phi - \frac{ig}{2}\tau^a W^a_\mu\phi - \frac{ig'}{2}B_\mu\phi. \tag{B-39}$$

Then:

$$\begin{aligned}(\mathrm{D}_\mu\phi^\dagger)(\mathrm{D}^\mu\phi) =&(\partial_\mu\phi^\dagger + \frac{ig}{2}\phi^\dagger\tau^a W^a_\mu + \frac{ig'}{2}\phi^\dagger B_\mu)\\ &\times(\partial^\mu\phi - \frac{ig}{2}\tau^b W^{b\mu}\phi - \frac{ig'}{2}B^\mu\phi)\\ =&(\partial_\mu\phi^\dagger)(\partial^\mu\phi) + \left|\frac{g}{2}\tau^a W^a_\mu\phi + \frac{g'}{2}B_\mu\phi\right|^2\\ &+ \text{(crossing term)}.\end{aligned} \tag{B-40}$$

In the last line, we have expand the covariant derivative of the scalar into three groups. The first group gives all the kinetic terms of various scalars. The second group gives all the vertices which do not involve partial derivatives. They contains all the 4 point interactions, and the 3 point interactions with no derivatives, and the mass term for W and Z gauge bosons. Finally, the third group, namely the crossing term, contains all the 3 point interactions with partial derivatives, as well as scalar-gauge mixing. As we shall see, this mixing involves $\pi^\pm$ and π^0 components of the scalar. Thus these components are not physical fields.

Our task now is to expand these three groups, to make the statements above explicit. The final expression should in such a form that all the compact notations, such as inner summation indices, disappear. The derivations are fairly lengthy, so be patient please.

The first group in (B-40) reads

$$(\partial_\mu\phi^\dagger)(\partial^\mu\phi) = \partial_\mu\pi^-\partial^\mu\pi^+ + \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0. \tag{B-41}$$

As claimed, they are kinetic terms of various kinds of scalars.

Next we move on to the second group of (B-40).

$$\begin{aligned}
&\left|\tfrac{g}{2}\tau^a W_\mu^a\phi + \tfrac{g'}{2}B_\mu\phi\right|^2\\
&=\underbrace{\tfrac{g^2}{4}W_\mu^a W^{b\mu}\phi^\dagger\tau^a\tau^b\phi}_{\text{2ndT(a)}}+\underbrace{\tfrac{g'^2}{4}B_\mu B^\mu\phi^\dagger\phi}_{\text{2ndT(b)}}+\underbrace{\tfrac{gg'}{2}W_\mu^a B^\mu\phi^\dagger\tau^a\phi}_{\text{2ndT(c)}}.
\end{aligned} \tag{B-42}$$

Three terms again. Be alert, we will expand these by brute force!

The first term in (B-42):

$$\begin{aligned}
&\tfrac{g^2}{4}W_\mu^a W^{b\mu}\phi^\dagger\tau^a\tau^b\phi\\
=&\tfrac{g^2}{4}W_\mu^a W^{b\mu}\phi^\dagger\tfrac{1}{2}(\tau^a\tau^b+\tau^b\tau^a)\phi=\tfrac{g^2}{4}W_\mu^a W^{a\mu}\phi^\dagger\phi\\
=&\tfrac{g^2}{4}(2W_\mu^+W^{-\mu}+W_\mu^3W^{3\mu})(\pi^+\pi^-+\tfrac{1}{2}(\pi^0)^2+\tfrac{1}{2}(h+v)^2),
\end{aligned} \tag{B-43}$$

In these steps We symmetrize the Pauli matrices, use the anticommutation relations

$$\tau^a\tau^b+\tau^b\tau^a=2\delta^{ab},$$

and use also the explicit form of W_μ^a and ϕ, namely, (B-26) and (B-22).

The second term in (B-42):

$$\begin{aligned}
&\tfrac{g'^2}{4}B_\mu B^\mu\phi^\dagger\phi\\
=&\tfrac{g'^2}{4}B_\mu B^\mu[\pi^+\pi^-+\tfrac{1}{2}(\pi^0)^2+\tfrac{1}{2}(h+v)^2],
\end{aligned} \tag{B-44}$$

The third term in (B-42):

$$\begin{aligned}
&\tfrac{gg'}{2}W_\mu^a B^\mu\phi^\dagger\tau^a\phi\\
=&\tfrac{gg'}{2}B^\mu\begin{pmatrix}\pi^- & \tfrac{1}{\sqrt{2}}(h+v-i\pi^0)\end{pmatrix}\\
&\times\begin{pmatrix}W_\mu^3 & W_\mu^1-iW_\mu^2\\ W_\mu^1+iW_\mu^2 & -W_\mu^3\end{pmatrix}\begin{pmatrix}\pi^+\\ \tfrac{1}{\sqrt{2}}(h+v+i\pi^0)\end{pmatrix}\\
=&\underbrace{\tfrac{gg'}{2}B^\mu W_\mu^3[\pi^+\pi^--\tfrac{1}{2}(\pi^0)^2-\tfrac{1}{2}(h+v)^2]}_{\text{2ndT(c1)}}\\
&+\underbrace{\tfrac{gg'}{2}B^\mu(W_\mu^+\pi^-+W_\mu^-\pi^+)(h+v)}_{\text{2ndT(c2)}}
\end{aligned}$$

$$+\underbrace{\tfrac{gg'}{2}B^\mu(W_\mu^+\pi^- - W_\mu^-\pi^+)i\pi^0}_{\text{2ndT(c3)}}. \tag{B-45}$$

Now we sum them up:

$$\begin{aligned}\text{2ndT(a)} + \text{2ndT(b)} + \text{2ndT(c1)} = &\underbrace{\tfrac{g^2}{2}W_\mu^+W^{-\mu}[\pi^+\pi^- + \tfrac{1}{2}(\pi^0)^2 + \tfrac{1}{2}(h+v)^2]}_{\text{2ndT(d)}}\\ &+\underbrace{\tfrac{1}{4}[g^2W_\mu^3W^{3\mu} + g'^2B_\mu B^\mu + 2gg'W_\mu^3B^\mu]\pi^+\pi^-}_{\text{2ndT(e)}}\\ &+\underbrace{\tfrac{1}{8}(g^2W_\mu^3W^{3\mu} + g'^2B_\mu B^\mu - 2gg'W_\mu^3B^\mu)[(h+v)^2 + (\pi^0)^2]}_{\text{2ndT(f)}}.\end{aligned} \tag{B-46}$$

We further expand the three terms on the right hand side:

$$\begin{aligned}\text{2ndT(d)} = &\tfrac{g^2}{2}W_\mu^+W^{-\mu}\pi^+\pi^- + \tfrac{g^2}{4}W_\mu^+W^{-\mu}\pi^0\pi^0 + \tfrac{g^2}{4}W_\mu^+W^{-\mu}h^2\\ &+\tfrac{g^2v}{2}W_\mu^+W^{-\mu}h + \tfrac{g^2v^2}{4}W_\mu^+W^{-\mu},\end{aligned} \tag{B-47}$$

$$\begin{aligned}\text{2ndT(e)} = &\tfrac{1}{4}\Big[g^2(c_WZ_\mu + s_WA_\mu)^2 + g'^2(-s_WZ_\mu + c_WA_\mu)^2\\ &+ 2gg'(c_WZ_\mu + s_WA_\mu)(-s_WZ_\mu + c_WA_\mu)\Big]\pi^+\pi^-\\ =&\tfrac{1}{4}\Big[\tfrac{(g^2-g'^2)^2}{g^2+g'^2}Z_\mu Z^\mu + \tfrac{4g^2g'^2}{g^2+g'^2}A_\mu A^\mu + \tfrac{4gg'(g^2-g'^2)}{g^2+g'^2}Z_\mu A^\mu\Big]\pi^+\pi^-\\ =&(g\tfrac{\cos 2\theta_W}{2\cos\theta_W})^2Z_\mu Z^\mu\pi^+\pi^- + e^2A_\mu A^\mu\pi^+\pi^- + (4g^2\tan\theta_W\cos 2\theta_W)Z_\mu A^\mu\pi^+\pi^-,\end{aligned} \tag{B-48}$$

$$\begin{aligned}\text{2ndT(f)} =&\tfrac{1}{8}\Big[g^2(c_WZ_\mu + s_WA_\mu)^2 + g'^2(-s_WZ_\mu + c_WA_\mu)^2\\ &- 2gg'(c_WZ_\mu + s_WA_\mu)(-s_WZ_\mu + c_WA_\mu)\Big][(h+v)^2 + (\pi^0)^2]\\ =&\tfrac{1}{4}[(g^2 + g'^2)Z_\mu Z^\mu][(h+v)^2 + (\pi^0)^2]\\ =&\Big(\tfrac{g}{2c_W}\Big)^2(Z_\mu Z^\mu h^2 + Z_\mu Z^\mu\pi^0\pi^0 + vZ_\mu Z^\mu h) + (\tfrac{gv}{2c_W})^2Z_\mu Z^\mu.\end{aligned} \tag{B-49}$$

Don't forget another two terms:

$$\begin{aligned}\text{2ndT(c2)} =&\tfrac{gg'}{2}(-s_WZ^\mu + c_WA^\mu)(W_\mu^+\pi^- + W_\mu^-\pi^+)(h+v)\\ =&\tfrac{-s_Wgg'}{2}(Z^\mu W_\mu^+\pi^-h + Z^\mu W_\mu^-\pi^+h) + \tfrac{c_Wgg'}{2}(A^\mu W_\mu^+\pi^-h + A^\mu W_\mu^-\pi^+h)\end{aligned}$$

$$+\frac{-s_W t_W g^2 v}{2}(Z^\mu W^+_\mu \pi^- + Z^\mu W^-_\mu \pi^+) + \frac{s_W g^2 v}{2}(A^\mu W^+_\mu \pi^- + A^\mu W^-_\mu \pi^+). \quad \text{(B-50)}$$

$$\begin{aligned}
\text{2ndT(c3)} &= \frac{gg'}{2}(-s_W Z^\mu + c_W A^\mu)(W^+_\mu \pi^- - W^-_\mu \pi^+) i\pi^0 \\
&= \frac{-i s_W t_W g^2}{2}(Z^\mu W^+_\mu \pi^- \pi^0 - Z^\mu W^-_\mu \pi^+ \pi^0) \quad \text{(B-51)} \\
&\quad + \frac{i s_W g^2}{2}(A^\mu W^+_\mu \pi^- \pi^0 - A^\mu W^-_\mu \pi^+ \pi^0). \quad \text{(B-52)}
\end{aligned}$$

Now, 2ndT(d), 2ndT(e), 2ndT(f), 2ndT(c2), 2ndT(c3) are readily in the form from which we can read the Feynman rules directly.

Going on expanding the ugly 3rdT:

$$\begin{aligned}
\text{3rdT} &= \Big(\frac{ig}{2}\tau^a W^{b\mu} + \frac{ig'}{2}B^\mu\Big)\Big(\phi^\dagger \partial_\mu \phi - (\partial_\mu \phi^\dagger)\phi\Big) \\
&= -\begin{pmatrix} \pi^- & \frac{1}{\sqrt{2}}(h + v - i\pi^0) \end{pmatrix} \mathbf{M}^\mu \begin{pmatrix} \partial_\mu \pi^+ \\ \frac{1}{\sqrt{2}}(\partial_\mu h + i\partial_\mu \pi^0) \end{pmatrix} \\
&= -\begin{pmatrix} \partial_\mu \pi^- & \frac{1}{\sqrt{2}}(\partial_\mu h - i\partial_\mu \pi^0) \end{pmatrix} \mathbf{M}^\mu \begin{pmatrix} \pi^+ \\ \frac{1}{\sqrt{2}}(h + v + i\pi^0) \end{pmatrix}, \quad \text{(B-53)}
\end{aligned}$$

where the matrix $\mathbf{M}_\mu$ is:

$$\mathbf{M}_\mu = \begin{pmatrix} \frac{i}{2}(gW^3_\mu + g'B_\mu) & \frac{ig}{2}(W^1_\mu - iW^2_\mu) \\ \frac{ig}{2}(W^1_\mu + iW^2_\mu) & \frac{i}{2}(-gW^3_\mu + g'B_\mu) \end{pmatrix} = \begin{pmatrix} \frac{i}{2}gc_W(1 - t_W^2)Z_\mu + ieA_\mu & \frac{ig}{\sqrt{2}}W^+_\mu \\ \frac{ig}{\sqrt{2}}W^-_\mu & -\frac{ig}{2c_W}Z_\mu \end{pmatrix}. \quad \text{(B-54)}$$

Thus:

$$\begin{aligned}
\text{3rdT} =&\Big\{\frac{igc_W(1-t_W^2)}{2}Z_\mu \pi^- \partial^\mu \pi^+ + ieA_\mu \pi^- \partial^\mu \pi^+ + \frac{ig}{2}W^+_\mu \pi^- \partial^\mu h - \frac{g}{2}W^+_\mu \pi^- \partial^\mu \pi^0 \\
&\frac{ig}{2}W^-_\mu h \partial^\mu \pi^+ + \frac{igv}{2}W^-_\mu \partial^\mu \pi^+ - \frac{g}{2}W^-_\mu \pi^0 \partial^\mu \pi^+ - \frac{ig}{4c_W}Z_\mu(h + v - i\pi^0)(\partial^\mu h + i\partial^\mu \pi^0)\Big\} \\
&-\Big\{\frac{igc_W(1-t_W^2)}{2}Z_\mu \pi^+ \partial^\mu \pi^- + ieA_\mu \pi^+ \partial^\mu \pi^- + \frac{ig}{2}W^-_\mu \pi^+ \partial^\mu h + \frac{g}{2}W^-_\mu \pi^+ \partial^\mu \pi^0 \\
&+ \frac{ig}{2}W^+_\mu \partial^\mu \pi^- h + \frac{igv}{2}W^+_\mu \partial^\mu \pi^- - \frac{g}{2}W^+_\mu \pi^0 \partial^\mu \pi^- - \frac{ig}{4c_W}Z_\mu(h + v + i\pi^0)(\partial_\mu h - i\partial_\mu \pi^0)\Big\} \\
=&\frac{igc_W(1-t_W^2)}{2}Z_\mu(\pi^- \partial^\mu \pi^+ - \pi^+ \partial^\mu \pi^-) + ieA_\mu(\pi^- \partial^\mu \pi^+ - \pi^+ \partial^\mu \pi^-) \\
&+ \frac{ig}{2}W^+_\mu(\pi^- \partial^\mu h - h\partial^\mu \pi^-) - \frac{g}{2}W^+_\mu(\pi^- \partial^\mu \pi^0 - \pi^0 \partial^\mu \pi^-) \\
&- \frac{ig}{2}W^-_\mu(\pi^+ \partial^\mu h - h\partial^\mu \pi^+) - \frac{g}{2}W^-_\mu(\pi^+ \partial_\mu \pi^0 - \pi^0 \partial_\mu \pi^+) \\
&+ \frac{igv}{2}(W^-_\mu \partial^\mu \pi^+ - W^+_\mu \partial^\mu \pi^-) + \frac{g}{2c_W}Z^\mu(h\partial_\mu \pi^0 - \pi^0 \partial_\mu h) + \frac{gv}{2c_W}Z^\mu \partial_\mu \pi^0. \quad \text{(B-55)}
\end{aligned}$$

This is the final results. Don't worry about the mixing terms like $W\partial\pi$, they will be eliminated by gauge fixing.

B.2.5 Gauge Fixing and the Ghost Field

As has been mentioned, the quantization of a non-Abelian gauge theory with Faddeev-Popov method will involve two additional terms in the classical Lagrangian. They are gauge fixing term and ghost term.

B.2.5.1 Gauge Fixing

The gauge fixing are chosen to be in the so-called R_ξ gauge. This gauge has the advantage that it can eliminate the gauge-higgs mixing appeared above.

$$\mathcal{L}_{\text{GF}} = -\tfrac{1}{2\xi}(\partial_\mu W^{a\mu} - \xi\kappa_a\chi^a)^2 - \tfrac{1}{2\xi}(\partial_\mu B^\mu - \xi\kappa_B\pi^0)^2. \tag{B-56}$$

where:

$$\chi = \begin{pmatrix} -\frac{i}{\sqrt{2}}(\pi^+ - \pi^-) \\ \frac{1}{\sqrt{2}}(\pi^+ + \pi^-) \\ \pi^0 \end{pmatrix}, \qquad \kappa_1 = \kappa_2 \equiv \kappa_W = -\frac{gv}{2}, \qquad \kappa_3 = \frac{gv}{2}, \qquad \kappa_B = \frac{gvt_W}{2}. \tag{B-57}$$

Then we have:

$$\begin{aligned} -\frac{1}{2\xi}\Big(\partial_\mu W^{a\mu} - \xi\kappa_a\chi^a\Big)^2 &= -\frac{1}{2\xi}(\partial_\mu W^{a\mu})^2 - \frac{1}{2\xi}(\xi\kappa_a\tilde{\phi}^a)^2 + \partial_\mu W^{a\mu}\kappa_a\tilde{\phi}^a \\ = -\frac{1}{\xi}\partial_\mu W^{+\mu}\partial_\nu W^{-\nu} &- \frac{1}{2\xi}(\partial_\mu W^{3\mu})^2 - \xi\kappa_W^2\pi^+\pi^- - \frac{\xi\kappa_3^2}{2}(\pi^0)^2 \\ + i\kappa_W\Big(\pi^-\partial^\mu W_\mu^+ - \pi^+\partial^\mu W_\mu^-\Big) &+ (c_W\partial_\mu Z^\mu + s_W\partial_\mu A^\mu)\kappa_3\pi^0, \end{aligned} \tag{B-58}$$

and:

$$-\frac{1}{2\xi}\Big(\partial_\mu B^\mu - \xi\kappa_B\phi^3\Big)^2 = -\frac{1}{2\xi}(\partial_\mu B^\mu)^2 - \frac{\xi\kappa_B^2(\pi^0)^2}{2} + \kappa_B(-s_W\partial^\mu Z_\mu + c_W\partial^\mu A_\mu)\pi^0. \tag{B-59}$$

Adding them together, we get:

$$\begin{aligned} \mathcal{L}_{\text{GF}} = &-\frac{1}{\xi}\partial_\mu W^{+\mu}\partial_\nu W^{-\nu} - \frac{1}{2\xi}(\partial_\mu W^{3\mu})^2 - \frac{1}{2\xi}(\partial_\mu B^\mu)^2 - \xi\kappa_W^2\pi^+\pi^- - \frac{\xi\kappa_3^2}{2c_W^2}(\pi^0)^2 \\ &+ i\kappa_W\Big(\pi^-\partial^\mu W_\mu^+ - \pi^+\partial^\mu W_\mu^-\Big) + \frac{\kappa_3}{c_W}\pi^0\partial_\mu Z^\mu. \end{aligned} \tag{B-60}$$

B.2.5.2 The Ghost Field

For convenience, we rewrite the gauge fixing lagrangian as:

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{\xi}F_{+}F_{-} - \frac{1}{2\xi}F_{Z}^{2} - \frac{1}{2\xi}F_{A}^{2}, \tag{B-61}$$

where:

$$F_{\pm} = \partial^{\mu}W_{\mu}^{+} \pm i\xi\kappa_{W}\pi^{\pm}, \tag{B-62}$$

$$F_{Z} = \partial^{\mu}Z_{\mu} - \xi\kappa_{Z}\pi^{0}, \tag{B-63}$$

$$F_{A} = \partial^{\mu}A_{\mu}. \tag{B-64}$$

Then the ghost terms can be written as:

$$\mathcal{L}_{\mathrm{ghost}} = \bar{c}^{a}\frac{\delta F^{a}}{\delta\theta^{b}}c^{b}. \tag{B-65}$$

Where $\theta_{a} = (\theta_{+}, \theta_{-}, \theta_{Z}, \theta_{A})$ are generators of the gauge group. Now we expand this term as follows:

$$\begin{aligned}\mathcal{L}_{\mathrm{ghost}} =& \bar{c}^{+}\frac{\delta F_{+}}{\delta\theta^{b}}c^{b} + \bar{c}^{-}\frac{\delta F_{-}}{\delta\theta^{b}}c^{b} + \bar{c}_{Z}\frac{\delta F_{Z}}{\delta\theta^{b}}c^{b} + \bar{c}_{A}\frac{\delta F_{A}}{\delta\theta^{b}}c^{b}\\ =& \bar{c}^{+}\Big(\partial^{\mu}\frac{\delta W_{\mu}^{+}}{\delta\theta^{b}} + i\xi\kappa_{W}\frac{\delta\pi^{+}}{\delta\theta^{b}}\Big)c^{b} + \bar{c}^{-}\Big(\partial^{\mu}\frac{\delta W_{\mu}^{-}}{\delta\theta^{b}} - i\xi\kappa_{W}\frac{\delta\pi^{-}}{\delta\theta^{b}}\Big)c^{b}\\ &+ \bar{c}_{Z}\Big(\partial^{\mu}\frac{\delta Z_{\mu}}{\delta\theta^{b}} - \xi\kappa_{Z}\frac{\delta\pi^{0}}{\delta\theta^{b}}\Big)c^{b} + \bar{c}_{A}\Big(\partial^{\mu}\frac{\delta A_{\mu}}{\delta\theta^{b}}\Big)c^{b}.\end{aligned} \tag{B-66}$$

Note that:

$$\delta W_{\mu}^{\pm} = (-\partial_{\mu} \pm ieA_{\mu} \pm igc_{W}Z_{\mu})\theta_{\pm} \mp igc_{W}W_{\mu}^{\pm}\theta_{Z} \mp ieW_{\mu}^{\pm}\theta_{A}; \tag{B-67}$$

$$\delta Z_{\mu} = -igc_{W}W_{\mu}^{-}\theta^{+} + igc_{W}W_{\mu}^{+}\theta^{-} - \partial_{\mu}\theta_{Z}; \tag{B-68}$$

$$\delta A_{\mu} = -ieW_{\mu}^{-}\theta^{+} + ieW_{\mu}^{+}\theta^{-} - \partial_{\mu}\theta_{A}; \tag{B-69}$$

$$\delta\pi^{\pm} = \Big(\mp \frac{ig}{2}(h+v) - \frac{g}{2}\pi^0 \Big)\theta^{\pm} \mp \frac{ig\cos\theta_W}{2c_W}\pi^{\pm}\theta_Z \mp ie\pi^{\pm}\theta_A; \tag{B-70}$$

$$\delta\pi^0 = \frac{g}{2}\pi^-\theta^+ - \frac{g}{2}\pi^+\theta^- + \frac{g}{2c_W}(h+v)\theta_Z. \tag{B-71}$$

From these relations we can deduce:

$$\begin{aligned}
\mathcal{L}_{\text{ghost}} =& \bar{c}^+\Big[\partial^\mu(-\partial_\mu + ieA_\mu + igc_W Z_\mu) + i\xi\kappa_W\Big(- \frac{ig}{2}(h+v) - \frac{g}{2}\pi^0\Big)\Big]c^+ \\
&+ \bar{c}^+\Big[\partial^\mu(-igc_W W^+_\mu) + i\xi\kappa_W\Big(- \frac{ig\cos\theta_W}{2c_W}\pi^+\Big)\Big]c_Z + \bar{c}^+\Big[\partial^\mu(-ieW^+_\mu) + i\xi\kappa_W(-ie\pi^+)\Big]c_A \\
&+ \bar{c}^-\Big[\partial^\mu(-\partial_\mu - ieA_\mu - igc_W Z_\mu) - i\xi\kappa_W\Big(\frac{ig}{2}(h+v) - \frac{g}{2}\pi^0\Big)\Big]c^- \\
&+ \bar{c}^-\Big[\partial^\mu(igc_W W^-_\mu) - i\xi\kappa_W\Big(\frac{ig\cos 2\theta_W}{2c_W}\pi^-\Big)\Big]c_Z + \bar{c}^-\Big[\partial^\mu(ieW^-_\mu) - i\xi\kappa_W(ie\pi^-)\Big]c_A \\
&+ \bar{c}_Z\Big[\partial^\mu(\mp igc_W W^\mp_\mu) - \xi\kappa_Z\Big(\pm\frac{g}{2}\pi^\mp\Big)\Big]c^\pm + \bar{c}_Z\Big[- \partial^\mu\partial_\mu - \xi\kappa_Z\Big(\frac{g}{2c_W}(h+v)\Big)\Big]c_Z \\
&+ \bar{c}_A\Big[\mp ie\partial^\mu W^\mp_\mu\Big]c^\pm + \bar{c}_A\Big[- \partial^\mu\partial_\mu\Big]c_A.
\end{aligned} \tag{B-72}$$

We write it into a more clean form:

$$\begin{aligned}
\mathcal{L}_{\text{ghost}} =& \bar{c}^+\Big(- \partial^2 - \xi M_W^2\Big)c^+ + \bar{c}^-\Big(- \partial^2 - \xi M_W^2\Big)c^- + \bar{c}_Z\Big(- \partial^2 - \xi M_Z^2\Big)c_Z + \bar{c}_A(-\partial^2)c_A \\
&+ \bar{c}^\pm\Big(\pm ie\partial^\mu A_\mu \pm igc_W\partial^\mu Z_\mu - \frac{\xi M_W^2}{v}h \pm \frac{i\xi M_W^2}{v}\pi^0\Big)c^\pm \\
&+ \bar{c}^\pm\Big(\mp igc_W\partial^\mu W^\pm_\mu \pm \frac{\xi M_W^2\cos 2\theta_W}{c_W v}\pi^\pm\Big)c_Z + \bar{c}_Z\Big(\mp igc_W\partial^\mu W^\mp_\mu \mp \frac{\xi g M_Z}{2}\pi^\mp\Big)c^\pm \\
&+ \bar{c}^\pm\Big(\mp ie\partial^\mu W^\pm_\mu \mp e\xi M_W\pi^\pm\Big)c_A + \bar{c}_A(\mp ie\partial^\mu W^\mp_\mu)c^\pm - \bar{c}_Z\frac{\xi M_Z^2}{v}hc_Z.
\end{aligned} \tag{B-73}$$

B.2.6 Fermion-Gauge Sector

The fermions in standard model are as follows:

Leptons	Rep. of $SU(2)$	Isospin3 I_3	Supercharge Y
ν_{eL} $\nu_{\mu L}$ $\nu_{\tau L}$	$2d$	$1/2$	$-1/2$
e_L^- μ_L^- τ_L^-	$2d$	$-1/2$	$-1/2$
e_R^- μ_R^- τ_R^-	$1d$	0	-1
u_L c_L t_L	$2d$	$1/2$	$1/6$
d_L s_L b_L	$2d$	$-1/2$	$1/6$
u_R c_R t_R	$1d$	0	$2/3$
d_R s_R b_R	$1d$	0	$-1/3$

In what follows, we will group the left-handed leptons into a group, called L_L, the right-handed leptons will be written as l_R. Accordingly, the left-handed quarks will be grouped as Q_L, and the right-handed quarks will be written as q_R^u and q_L^d.

In this manner, we can write the Lagrangian for fermions as:

$$\mathcal{L}_{\mathrm{F}} = \bar{L}_L \not{D}_{lL} L_L + \bar{l}_R \not{D}_{lR} l_R + \bar{Q}_L \not{D}_{qL} Q_L + \bar{q}_R^u \not{D}_{uR} q_R^u + \bar{q}_R^d \not{D}_{dR} q_R^d. \tag{B-74}$$

We will expand this formula term by term. Before going into the detail, we note a left-handed fermion ψ_L and a right-handed fermion ψ_R can be expressed by a corresponding projection operators P_L and P_R acting on a Dirac spinor ψ, as:

$$\psi_L = P_L \psi = \frac{1-\gamma^5}{2}\psi, \qquad \psi_R = P_R \psi = \frac{1+\gamma^5}{2}\psi. \tag{B-75}$$

These relations are very useful when we combine the Lagrangians of left-handed and right-handed parts.

B.2.6.1 The Left-handed Leptons

As has been mentioned, we have:

$$L_L = \begin{pmatrix} \nu_i \\ e_i^- \end{pmatrix}_L, \qquad i = e, \mu, \tau. \tag{B-76}$$

The corresponding covariant derivative is:

$$\mathrm{D}_{lL}^{\mu} = i\partial^{\mu} + \frac{g}{2}\tau^a W^{a\mu} + g' Y_{lL} B^{\mu}. \tag{B-77}$$

Remember that $Y_{lL} = -1/2$, thus we have:

$$\Rightarrow \quad \mathrm{D}_{1L}^{\mu} = \begin{pmatrix} i\partial^{\mu} + \frac{g}{2}W^{3\mu} - \frac{g'}{2}B^{\mu} & \frac{g}{\sqrt{2}}W^{+\mu} \\ \frac{g}{\sqrt{2}}W^{-\mu} & i\partial^{\mu} - \frac{g}{2}W^{3\mu} - \frac{g'}{2}B^{\mu} \end{pmatrix}$$

$$=\begin{pmatrix} i\partial^\mu + \frac{g}{2c_W}Z^\mu & \frac{g}{\sqrt{2}}W^{+\mu} \\ \frac{g}{\sqrt{2}}W^{-\mu} & i\partial^\mu - \frac{gc_W(1-t_W^2)}{2}Z^\mu - eA^\mu \end{pmatrix}. \tag{B-78}$$

Then:

$$\begin{aligned}\bar{L}_L \not{D}_{lL} L_L =& \bar{\nu}_{Li} i\not{\partial}\nu_{Li} + e^+_{Li} i\not{\partial} e^-_{Li} + \frac{g}{2c_W}\bar{\nu}_{Li}\not{Z}\nu_{Li} - \frac{gc_W(1-t_W^2)}{2}e^+_{Li}\not{Z}e^-_{Li} - ee^+_{Li}\not{A}e^-_{Li} \\ &+ \frac{g}{\sqrt{2}}\bar{\nu}_{Li}\not{W}^+ e^-_{Li} + \frac{g}{\sqrt{2}}e^+_{Li}\not{W}^-\nu_{Li}.\end{aligned} \tag{B-79}$$

B.2.6.2 The Right-handed Leptons

Now we turn to the right-handed leptons. Note that there are no right-handed neutrinos, hence:

$$l_R = e^-_{Ri}, \qquad i = e, \mu, \tau. \tag{B-80}$$

The covariant derivative:

$$\mathrm{D}^\mu_{lR} = i\partial^\mu + g' Y_{lR} B^\mu = i\partial^\mu + g' B^\mu = i\partial^\mu - g t_W s_W Z^\mu + eA^\mu. \tag{B-81}$$

Thus:

$$\bar{l}_R \not{D}_{lR} l_R = e^+_{Ri} i\not{\partial} e^-_{Ri} + g t_W s_W e^+_{Ri}\not{Z}e^-_{Ri} - ee^+_{Ri}\not{A}e^-_{Ri}. \tag{B-82}$$

Now we can combine the two terms involving leptons by introducing γ^5 (Recall the projection operators mentioned above):

$$\begin{aligned}\bar{L}_L \not{D}_{lL} L_L + \bar{l}_R \not{D}_{lR} l_R =& \bar{\nu}_{Li} i\not{\partial}\nu_{Li} + e^+_i i\not{\partial} e^-_i + \frac{g}{4c_W}\bar{\nu}_i \not{Z}(1-\gamma^5)\nu_i \\ &- \frac{gc_W(1-t_W^2)}{4}e^+_i\not{Z}(1-\gamma^5)e^-_i + \frac{g t_W s_W}{2}e^+_i\not{Z}(1+\gamma^5)e^-_i - ee^+_i\not{A}e^-_i \\ &+ \frac{g}{2\sqrt{2}}\bar{\nu}_i\not{W}^+(1-\gamma^5)e^-_i + \frac{g}{2\sqrt{2}}e^+_i\not{W}^-(1-\gamma^5)\nu_i \\ =& \bar{\nu}_{Li} i\not{\partial}\nu_{Li} + e^+_i i\not{\partial} e^-_i + \frac{g}{4c_W}\bar{\nu}_i\not{Z}(1-\gamma^5)\nu_i + \frac{g}{4}e^+_i\not{Z}\Big[c_W(3t_W^2-1) - \frac{\gamma^5}{c_W}\Big]e^-_i \\ &+ \frac{g}{2\sqrt{2}}\bar{\nu}_i\not{W}^+(1-\gamma^5)e^-_i + \frac{g}{2\sqrt{2}}e^+_i\not{W}^-(1-\gamma^5)\nu_i.\end{aligned} \tag{B-83}$$

B.2.6.3 The Left-handed Quarks

We can write out everything without further explanation:

$$Q_L = \begin{pmatrix} q^u_{Li} \\ q^d_{Li} \end{pmatrix} = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}. \tag{B-84}$$

The covariant derivative:

$$\begin{aligned} \mathrm{D}^\mu_{lL} =& i\partial^\mu + \frac{g}{2}\tau^a W^{a\mu} + g' Y_{qL} B^\mu \\ =& \begin{pmatrix} i\partial^\mu + \frac{g}{2}W^{3\mu} + \frac{g'}{6}B^\mu & \frac{g}{\sqrt{2}}W^{+\mu} \\ \frac{g}{\sqrt{2}}W^{-\mu} & i\partial^\mu - \frac{g}{2}W^{3\mu} + \frac{g'}{6}B^\mu \end{pmatrix} \\ =& \begin{pmatrix} i\partial^\mu + \frac{gc_W(1-t_W^2/3)}{2}Z^\mu + \frac{2}{3}eA^\mu & \frac{g}{\sqrt{2}}W^{+\mu} \\ \frac{g}{\sqrt{2}}W^{-\mu} & i\partial^\mu - \frac{gc_W(1+t_W^2/3)}{2}Z^\mu - \frac{1}{3}eA^\mu \end{pmatrix}. \end{aligned} \tag{B-85}$$

$$\begin{aligned} \bar{Q}_L \not{D}_{qL} Q_L =& \bar{q}^u_{Li} i\not{\partial} q^u_{Li} + \bar{q}^d_{Li} i\not{\partial} q^d_{Li} \\ &+ \frac{gc_W(1-t_W^2/3)}{2}\bar{q}^u_{Li}\not{Z} q^u_{Li} + \frac{2}{3}e\bar{q}^u_{Li}\not{A} q^u_{Li} - \frac{gc_W(1+t_W^2/3)}{2}\bar{q}^d_{Li}\not{Z} q^d_{Li} - \frac{1}{3}e\bar{q}^d_{Li}\not{A} q^d_{Li} \\ &+ \frac{g}{\sqrt{2}}\bar{q}^u_{Li}\not{W}^+ q^d_{Li} + \frac{g}{\sqrt{2}}\bar{q}^d_{Li}\not{W}^- q^u_{Li}. \end{aligned} \tag{B-86}$$

B.2.6.4 The Right-handed Quarks

The last part:

$$q^u_R = (q^u_{Ri}) = (u_R \quad c_R \quad t_R), \qquad q^d_R = (q^d_{Ri}) = (d_R \quad s_R \quad b_R). \tag{B-87}$$

The covariant derivatives:

$$\mathrm{D}^\mu_{uR} = i\partial^\mu + g' Y_{uR} B^\mu = i\partial^\mu + \frac{2}{3}g' B^\mu = i\partial^\mu - \frac{2gs_W^2}{3c_W}Z^\mu + \frac{2}{3}eA^\mu, \tag{B-88}$$

$$\mathrm{D}^\mu_{dR} = i\partial^\mu + g' Y_{dR} B^\mu = i\partial^\mu - \frac{1}{3}g' B^\mu = i\partial^\mu + \frac{gs_W^2}{3c_W}Z^\mu - \frac{1}{3}eA^\mu. \tag{B-89}$$

$$\bar{q}^u_R \not{D}_{uR} q^u_R = \bar{q}^u_{Ri} i\not{\partial} q^u_{Ri} - \frac{2gs_W^2}{3c_W}\bar{q}^u_{Ri}\not{Z} q^u_{Ri} + \frac{2}{3}e\bar{q}^u_{Ri}\not{A} q^u_{Ri}, \tag{B-90}$$

$$\bar{q}^d_R \not{D}_{uR} q^d_R = \bar{q}^d_{Ri} i\not{\partial} q^d_{Ri} + \frac{gs_W^2}{3c_W}\bar{q}^d_{Ri}\not{Z} q^d_{Ri} - \frac{1}{3}e\bar{q}^d_{Ri}\not{A} q^d_{Ri}. \tag{B-91}$$

We can also combine the left-handed and right-handed quarks by introducing γ^5, as in the case of leptons:

$$
\begin{aligned}
&\bar{Q}_L \not{D}_{qL} Q_L + \bar{q}^u_R \not{D}_{uR} q^u_R + \bar{q}^d_R \not{D}_{dR} q^d_R \\
=& \bar{q}^u_i i\not{\partial} q^u_i + \bar{q}^d_i i\not{\partial} q^d_i + \frac{g c_W (1 - t_W^2/3)}{4} \bar{q}^u_i \not{Z}(1-\gamma^5) q^u_i - \frac{g s_W^2}{3 c_W} \bar{q}^u_i \not{Z}(1+\gamma^5) q^u_i \\
&- \frac{g c_W (1 + t_W^2/3)}{4} \bar{q}^d_i \not{Z}(1-\gamma^5) q^d_i + \frac{g s_W^2}{6 c_W} \bar{q}^d_i \not{Z}(1+\gamma^5) q^d_i + \frac{2}{3} e \bar{q}^u_i \not{A} q^u_i - \frac{1}{3} e \bar{q}^d_i \not{A} q^d_i \\
&+ \frac{g}{2\sqrt{2}} \bar{q}^u_i \not{W}^+ (1-\gamma^5) q^d_i + \frac{g}{2\sqrt{2}} \bar{q}^d_i \not{W}^- (1-\gamma^5) q^u_i \\
=& \bar{q}^u_i i\not{\partial} q^u_i + \bar{q}^d_i i\not{\partial} q^d_i + \frac{g}{4} \bar{q}^u_i \not{Z}(c_W(1 - 5t_W^2/3) - \gamma^5/c_W) q^u_i + \frac{g}{4} \bar{q}^d_i \not{Z}(c_W(t_W^2/3 - 1) + \gamma^5/c_W) q^d_i \\
&+ \frac{2}{3} e \bar{q}^u_i \not{A} q^u_i - \frac{1}{3} e \bar{q}^d_i \not{A} q^d_i + \frac{g}{2\sqrt{2}} \bar{q}^u_i \not{W}^+ (1-\gamma^5) q^d_i + \frac{g}{2\sqrt{2}} \bar{q}^d_i \not{W}^- (1-\gamma^5) q^u_i . \qquad \text{(B-92)}
\end{aligned}
$$

B.2.7 Fermion-Higgs Sector

Now we turn to the last part, namely the Fermion-Higgs interaction. This interaction is of Yukawa type, and is the origin of the mass of various kinds of fermions. We must be very careful here, because the weak-coupling eigenstates and the mass eigenstates do not coincide, for both leptons and quarks. Thus we have to rotate the weak eigenbasis to the mass eigenbasis, by a unitary mapping acting on leptons or quarks.

The Lagrangian of this part can be written as:

$$
\mathcal{L}_{\text{FH}} = -y^l_{ij} \Big(\bar{L}_{Li} H l_{Rj} + \bar{l}_{Rj} H^\dagger L_{Li} \Big) - \Big(y^u_{ij} \bar{Q}_{Li} (i\tau_2 H^*) q^u_{Rj} + y^d_{ij} \bar{Q}_{Li} H q^d_{Rj} + \text{h.c.} \Big). \qquad \text{(B-93)}
$$

The subtlety we have mentioned are expressed here by the non-diagonal nature of various of y_{ij} matrices.

B.2.7.1 Lepton-Higgs Sector

$$
\mathcal{L}_{\text{LH}} = -\pi^+ \begin{pmatrix} \bar{\nu}_e & \bar{\nu}_\mu & \bar{\nu}_\tau \end{pmatrix} \begin{pmatrix} y^l_{11} & y^l_{12} & y^l_{13} \\ y^l_{21} & y^l_{22} & y^l_{23} \\ y^l_{31} & y^l_{32} & y^l_{33} \end{pmatrix} \begin{pmatrix} e^-_R \\ \mu^-_R \\ \tau^-_R \end{pmatrix} + \text{h.c.}
$$

$$
-\frac{1}{\sqrt{2}}(h+v+i\pi^0)\begin{pmatrix} e_L^+ & \mu_L^+ & \tau_L^+ \end{pmatrix}\begin{pmatrix} y_{11}^l & y_{12}^l & y_{13}^l \\ y_{21}^l & y_{22}^l & y_{23}^l \\ y_{31}^l & y_{32}^l & y_{33}^l \end{pmatrix}\begin{pmatrix} e_R^- \\ \mu_R^- \\ \tau_R^- \end{pmatrix} + \text{h.c.}. \qquad \text{(B-94)}
$$

Now we rotate the leptons into the basis in which the matrix y_{ij}^l gets diagonalized.

$$
\begin{pmatrix} e_L^+ & \mu_L^+ & \tau_L^+ \end{pmatrix}\begin{pmatrix} y_{11}^l & y_{12}^l & y_{13}^l \\ y_{21}^l & y_{22}^l & y_{23}^l \\ y_{31}^l & y_{32}^l & y_{33}^l \end{pmatrix}\begin{pmatrix} e_R^- \\ \mu_R^- \\ \tau_R^- \end{pmatrix} = \begin{pmatrix} e_L^{m+} & \mu_L^{m+} & \tau_L^{m+} \end{pmatrix}\begin{pmatrix} y_1^l & & \\ & y_2^l & \\ & & y_3^l \end{pmatrix}\begin{pmatrix} e_R^{m-} \\ \mu_R^{m-} \\ \tau_R^{m-} \end{pmatrix}. \qquad \text{(B-95)}
$$

Here and in the following, we use an upper index ”m” to label the mass eigenstate.

Then the second line of $\mathcal{L}_{LH}$ is:

$$
\begin{aligned}
&-\frac{1}{\sqrt{2}}(h+v+i\pi^0)\Big(y_1^l e_L^{m+} e_R^{m-} + y_2^l \mu_L^{m+} \mu_R^{m-} + y_2^l \tau_L^{m+} \tau_R^{m-}\Big) + \text{h.c.} \\
=&-\frac{1}{\sqrt{2}}(h+v)\Big(y_1^l e^{m+} e^{m-} + y_2^l \mu^{m+} \mu^{m-} + y_2^l \tau^{m+} \tau^{m-}\Big) \\
&-\frac{i}{\sqrt{2}}\pi^0\Big(y_1^l e^{m+}\gamma^5 e^{m-} + y_2^l \mu^{m+}\gamma^5 \mu^{m-} + y_2^l \tau^{m+}\gamma^5 \tau^{m-}\Big) \qquad \text{(B-96)}
\end{aligned}
$$

B.2.7.2 Quark-Higgs Sector

$$
\begin{aligned}
\mathcal{L}_{\text{QH}} =& -\frac{1}{\sqrt{2}}(h+v-i\pi^0)\begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix}(y_{ij}^u)\begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + \pi^-\begin{pmatrix} \bar{d}_L & \bar{s}_L & \bar{b}_L \end{pmatrix}(y_{ij}^u)\begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} \\
&-\frac{1}{\sqrt{2}}(h+v+i\pi^0)\begin{pmatrix} \bar{d}_L & \bar{s}_L & \bar{b}_L \end{pmatrix}(y_{ij}^d)\begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - \pi^+\begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix}(y_{ij}^d)\begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} + \text{h.c.}
\end{aligned}
$$

(B-97)

As in the case of leptons, we diagonalize the matrices y_{ij}^u and y_{ij}^d.

$$
\begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix}\begin{pmatrix} y_{11}^u & y_{12}^u & y_{13}^u \\ y_{21}^u & y_{22}^u & y_{23}^u \\ y_{31}^u & y_{32}^u & y_{33}^u \end{pmatrix}\begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} = \begin{pmatrix} \bar{u}_L^m & \bar{c}_L^m & \bar{t}_L^m \end{pmatrix}\begin{pmatrix} y_1^u & & \\ & y_2^u & \\ & & y_3^u \end{pmatrix}\begin{pmatrix} u_R^m \\ c_R^m \\ t_R^m \end{pmatrix} \qquad \text{(B-98)}
$$

$$\begin{pmatrix}\bar{d}_L & \bar{s}_L & \bar{b}_L\end{pmatrix}\begin{pmatrix}y^d_{11} & y^d_{12} & y^d_{13}\\ y^d_{21} & y^d_{22} & y^d_{23}\\ y^d_{31} & y^d_{32} & y^d_{33}\end{pmatrix}\begin{pmatrix}d_R\\ s_R\\ b_R\end{pmatrix} = \begin{pmatrix}\bar{d}^m_L & \bar{s}^m_L & \bar{b}^m_L\end{pmatrix}\begin{pmatrix}y^d_1 & & \\ & y^d_2 & \\ & & y^d_3\end{pmatrix}\begin{pmatrix}d^m_R\\ s^m_R\\ b^m_R\end{pmatrix} \tag{B-99}$$

Thus the first terms in each line of $\mathcal{L}_{\text{QH}}$, together with their hermitian conjugations, sum to:

$$\begin{aligned}&-\frac{1}{\sqrt{2}}(h+v)\Big(y^u_1\bar{u}^m u^m + y^u_2\bar{c}^m c^m + y^u_3\bar{t}^m t^m + y^d_1\bar{d}^m d^m + y^d_2\bar{s}^m s^m + y^d_3\bar{b}^m b^m\Big)\\ &+\frac{i\pi^0}{\sqrt{2}}\Big(y^u_1\bar{u}^m\gamma^5 u^m + y^u_2\bar{c}^m\gamma^5 c^m + y^u_3\bar{t}^m\gamma^5 t^m - y^d_1\bar{d}^m\gamma^5 d^m - y^d_2\bar{s}^m\gamma^5 s^m - y^d_3\bar{b}^m\gamma^5 b^m\Big)\end{aligned} \tag{B-100}$$

The remaining terms, namely the last term in each line of $\mathcal{L}_{\text{QH}}$, can also be expressed in the mass eigenbasis. The complexity comes from the fact that the unitary transformations from the weak basis to the mass basis are different for (u, c, t) quarks and (d, s, b) quarks. This difference is described by the famous CKM matrix:

$$V_{\text{CKM}} \equiv U^\dagger_{uL} U_{dL}, \tag{B-101}$$

where the unitary operator U_{uL} and U_{dL} are defined by:

$$U_{uL} q^m_{uL} = q_{uL}, \qquad U_{dL} q^m_{dL} = q_{dL}. \tag{B-102}$$

With the help of these formulae, we can rewrite the remaining terms in mass basis. Before doing this, we note that:

$$y^u_i \delta_{ij} = (U^\dagger_{uL})_{ik}(y^u)_{kl}(U_{uR})_{lj} \quad \Rightarrow \quad (y^u)_{mn} = y^u_i (U_{uL})_{mi}(U^\dagger_{uR})_{in}, \tag{B-103}$$

$$(U^\dagger_{dL})_{im}(y^u)_{mn}(U_{uR})_{nj} = y^u_k (U^\dagger_{dL})_{im}(U_{uL})_{mk}(U^\dagger_{uR})_{kn}(U_{uR})_{nj} = y^u_j (V^\dagger_{\text{CKM}})_{ij}, \tag{B-104}$$

and:

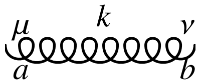
$$(U^\dagger_{uL})_{im}(y^d)_{mn}(U_{dR})_{nj} = y^d_k (U^\dagger_{uL})_{im}(U_{dL})_{mk}(U^\dagger_{dR})_{kn}(U_{dR})_{nj} = y^d_j (V_{\text{CKM}})_{ij}. \tag{B-105}$$

Now we are ready to write the needed terms:

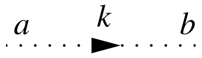
$$\pi^- \begin{pmatrix} \bar{d}_L^m & \bar{s}_L^m & \bar{b}_L^m \end{pmatrix} V_{\text{CKM}}^\dagger \begin{pmatrix} y_1^u u_R^m \\ y_2^u c_R^m \\ y_3^u t_R^m \end{pmatrix} - \pi^+ \begin{pmatrix} \bar{u}_L^m & \bar{c}_L^m & \bar{t}_L^m \end{pmatrix} V_{\text{CKM}} \begin{pmatrix} y_1^d d_R^m \\ y_2^d s_R^m \\ y_3^d b_R^m \end{pmatrix} + \text{h.c.} \qquad \text{(B-106)}$$

B.3 The Feynman Diagrams for QCD

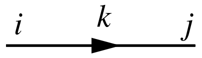
B.3.1 Propagators

G : 

$$= \frac{-i\delta_{ab}}{k^2 + i\epsilon}\Big[g^{\mu\nu} - (1 - \xi_C)\frac{k^\mu k^\nu}{k^2}\Big]$$

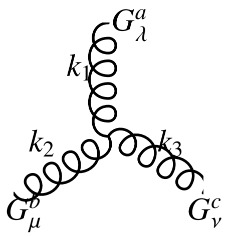
c_G : 

$$= \frac{i\delta_{ab}}{k^2 + i\epsilon}$$

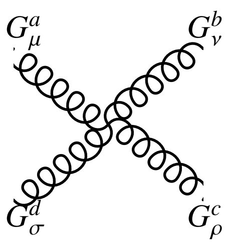
ψ : 

$$= \frac{i\delta_{ij}}{\not{k} - M_i + i\epsilon}$$

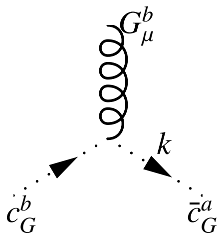
B.3.2 Vertices



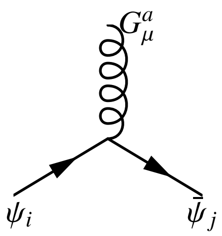
$$= g_C c^{abc}\Big(g^{\nu\lambda}(k_3 - k_1)^\mu + g^{\lambda\mu}(k_1 - k_2)^\nu + g^{\mu\nu}(k_2 - k_3)^\lambda\Big)$$



$$= -ig_C^2\Big(c^{eab}e^{ecd}(g^{\rho\mu}g^{\sigma\nu} - g^{\mu\sigma}g^{\nu\rho}) + c^{eac}c^{edb}(g^{\mu\sigma}g^{\rho\nu} - g^{\mu\nu}g^{\rho\sigma}) + c^{ead}e^{ebc}(g^{\mu\nu}g^{\rho\sigma} - g^{\rho\mu}g^{\sigma\nu})\Big)$$



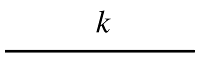
$$= -g_C c^{abc} k^\mu$$



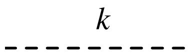
$$= i g_C t^a \gamma^\mu \delta_{ij}$$

B.4 Diagrams for Electroweak Theory

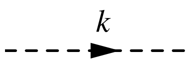
B.4.1 Propagators

h : 

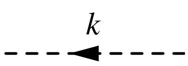
$$= \frac{i}{k^2 - 2\lambda v^2 + i\epsilon}$$

π^0 : 

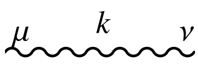
$$= \frac{i}{k^2 - \xi M_Z^2 + i\epsilon}$$

π^+ : 

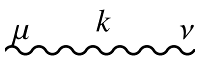
$$= \frac{i}{k^2 - \xi M_W^2 + i\epsilon}$$

π^- : 

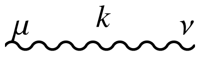
$$= \frac{i}{k^2 - \xi M_W^2 + i\epsilon}$$

W^+ : 

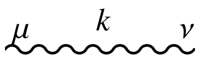
$$= \frac{-i}{k^2 - M_W^2 + i\epsilon}\Big[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_W^2}\Big]$$

W^- : 

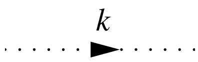
$$= \frac{-i}{k^2 - M_W^2 + i\epsilon}\Big[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_W^2}\Big]$$

$Z:$ 

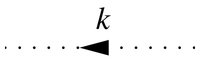
$$= \frac{-i}{k^2 - M_Z^2 + i\epsilon}\Big[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_Z^2}\Big]$$

$A:$ 

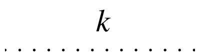
$$= \frac{-i}{k^2 + i\epsilon}\Big[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\Big]$$

$c^+:$ 

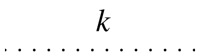
$$= \frac{i}{k^2 - \xi M_W^2 + i\epsilon}$$

$c^-:$ 

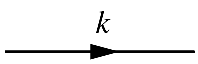
$$= \frac{i}{k^2 - \xi M_W^2 + i\epsilon}$$

$c_Z:$ 

$$= \frac{i}{k^2 - \xi M_Z^2 + i\epsilon}$$

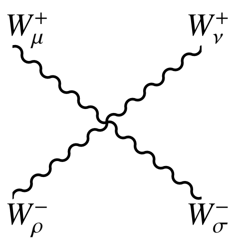
$c_A:$ 

$$= \frac{i}{k^2 + i\epsilon}$$

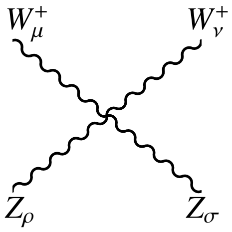
$f:$ 

$$= \frac{i}{\not{k} - M_f + i\epsilon}$$

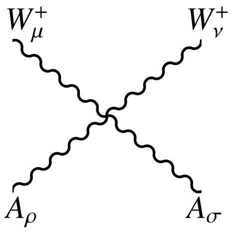
B.4.2 Pure Gauge Interactions



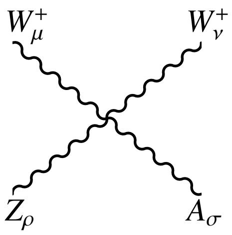
$$= ig^2(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})$$



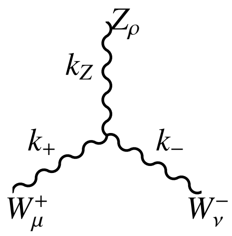
$$= -ig^2 c_W^2 (2g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$



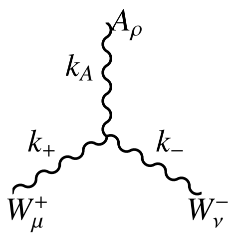
$$= -ie^2 (2g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$



$$= -iegc_W (2g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$



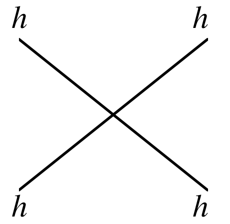
$$\begin{aligned} = &- igc_W [g^{\mu\nu}(k_+ - k_-)^\rho + g^{\nu\rho}(k_- - k_Z)^\mu \\ &+ g^{\rho\mu}(k_Z - k_+)^\nu] \end{aligned}$$



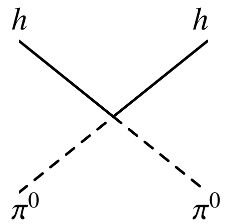
$$\begin{aligned} = &- ie\big(g^{\mu\nu}(k_+ - k_-)^\rho + g^{\nu\rho}(k_- - k_A)^\mu \\ &+ g^{\rho\mu}(k_A - k_+)^\nu\big) \end{aligned}$$

B.4.3 Higgs Self-Interactions

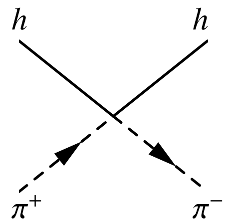
The following nine diagrams follow from the eqn.(B-38)



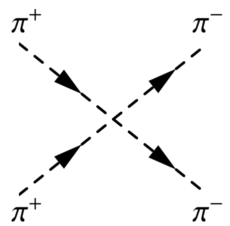
$= -6i\lambda$



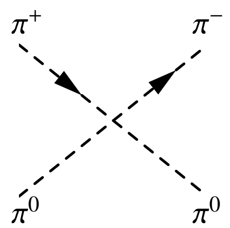
$= -2i\lambda$



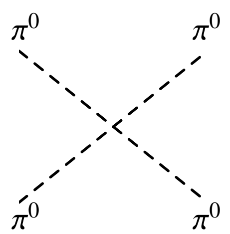
$= -2i\lambda$



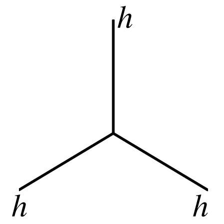
$= -4i\lambda$



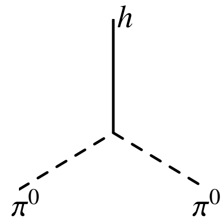
$= -2i\lambda$



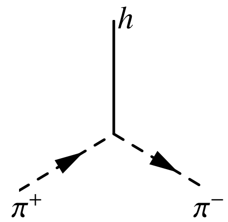
$= -6i\lambda$



$$= -6i\lambda v$$



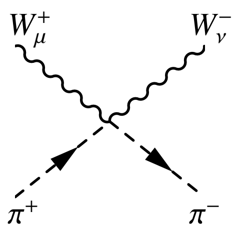
$$= -2i\lambda v$$



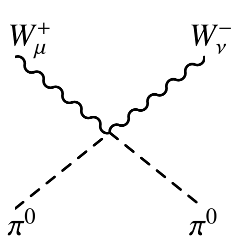
$$= -2i\lambda v$$

B.4.4 Higgs-Gauge Interactions

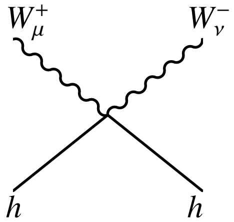
The following 16 diagrams of 2Higgs-2gauge bosons interactions follow from the Eqns.(B-47), (B-48), (B-49), (B-50), and (B-51).



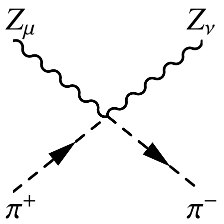
$$= \frac{ig^2}{2} g^{\mu\nu}$$



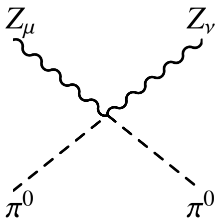
$$= \frac{ig^2}{2} g^{\mu\nu}$$



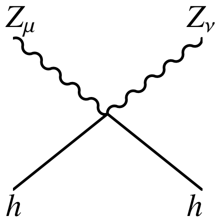
$$= \frac{ig^2}{2} g^{\mu\nu}$$



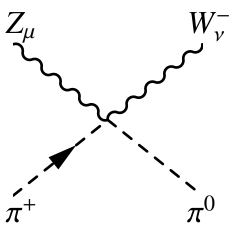
$$= \frac{ig^2 \cos^2 2\theta_W}{4c_W^2} g^{\mu\nu}$$



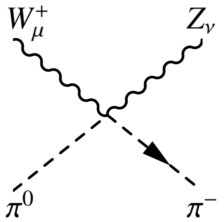
$$= \frac{ig^2}{2c_W^2} g^{\mu\nu}$$



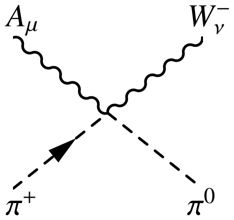
$$= \frac{ig^2}{2c_W^2} g^{\mu\nu}$$



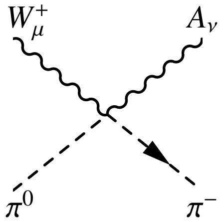
$$= -\frac{g^{\mu\nu} s_W t_W g^2}{2}$$



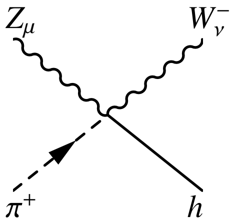
$$= \frac{g^{\mu\nu} s_W t_W g^2}{2}$$



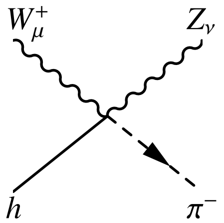
$$= \frac{g^{\mu\nu} s_W g^2}{2}$$



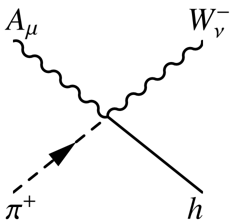
$$= -\frac{g^{\mu\nu} s_W g^2}{2}$$



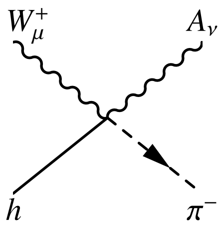
$$= -\frac{i g^{\mu\nu} s_W t_W g^2}{2}$$



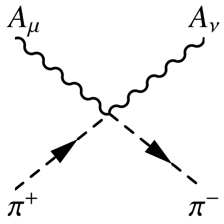
$$= -\frac{i g^{\mu\nu} s_W t_W g^2}{2}$$



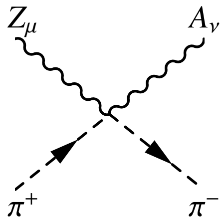
$$= \frac{i g^{\mu\nu} c_W t_W g^2}{2}$$



$$= \frac{i g^{\mu\nu} c_W t_W g^2}{2}$$

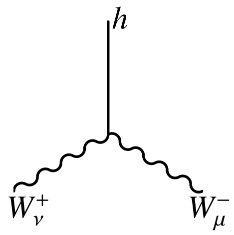


$$= ie^2 g^{\mu\nu}$$

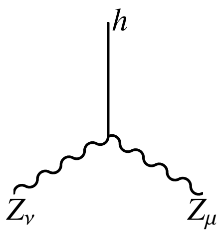


$$= 4ig^{\mu\nu} g^2 t_W \cos 2\theta_W$$

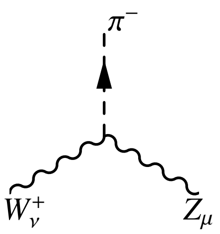
The following 13 diagrams of 1Higgs-2gauge bosons interactions follow from Eqn.(B-55).



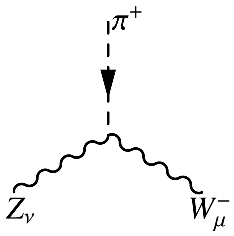
$$= \frac{ig^{\mu\nu} g^2 v}{2}$$



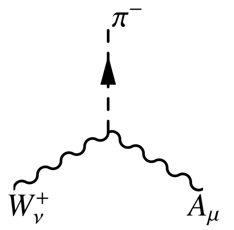
$$= \frac{ig^{\mu\nu} g^2 v}{4c_W^2}$$



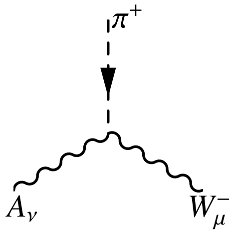
$$= -\frac{ig^{\mu\nu} s_W t_W g^2 v}{2}$$



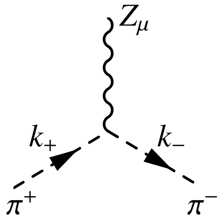
$$= -\frac{ig^{\mu\nu} s_W t_W g^2 v}{2}$$



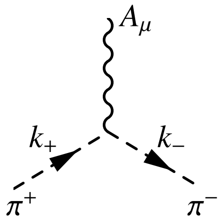
$$= \frac{ig^{\mu\nu} s_W g^2 v}{2}$$



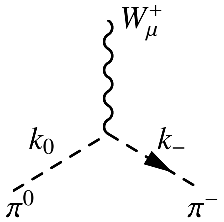
$$= \frac{ig^{\mu\nu} s_W g^2 v}{2}$$



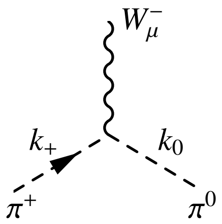
$$= -\frac{igc_W(1 - t_W^2)}{2}(k_+ - k_-)^\mu$$



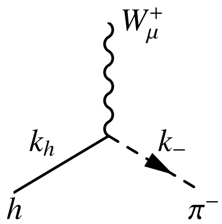
$$= -ie(k_+ - k_-)^\mu$$



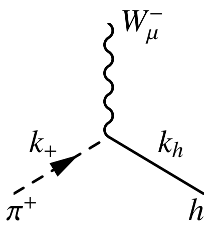
$$= \frac{g}{2}(k_0 - k_-)^\mu$$



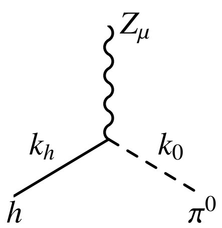
$$= \frac{g}{2}(k_0 - k_+)^\mu$$



$$= -\frac{ig}{2}(k_h - k_-)^\mu$$



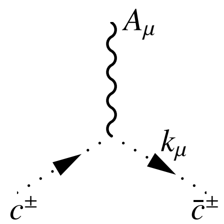
$$= -\frac{ig}{2}(k_+ - k_h)^\mu$$



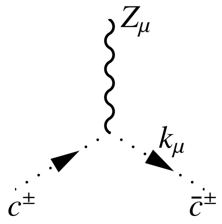
$$= \frac{g}{2c_W}(k_h - k_0)^\mu$$

B.4.5 Vertices with Ghost

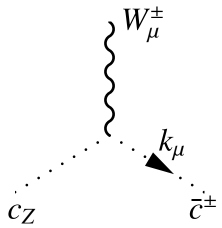
The following 12 diagrams with ghosts come from Eqn.(B-73).



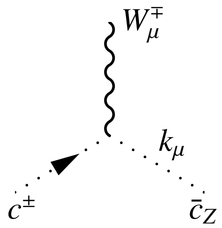
$$= \pm iek^\mu$$



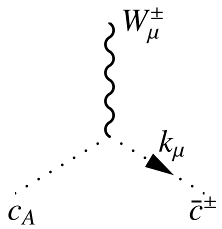
$$= \pm i g c_W k^\mu$$



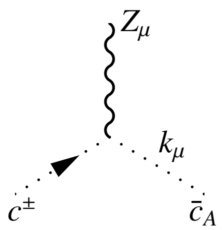
$$= \mp i g c_W k^\mu$$



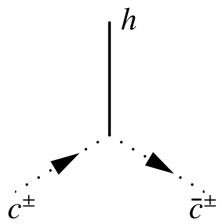
$$= \mp i g c_W k^\mu$$



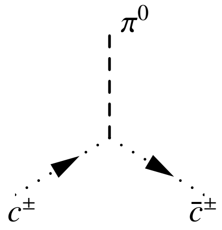
$$= \mp i e k^\mu$$



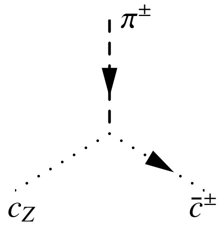
$$= \mp i e k^\mu$$



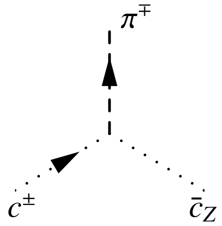
$$= -\frac{i\xi M_W^2}{v}$$



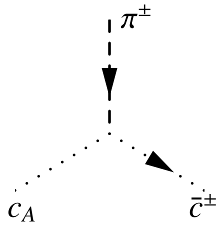
$$= \mp \frac{\xi M_W^2}{v}$$



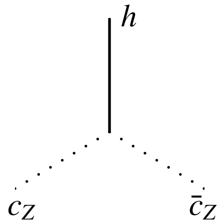
$$= \mp \frac{i\xi M_W^2 \cos 2\theta_W}{v c_W}$$



$$= \mp \frac{i\xi g M_Z}{2}$$

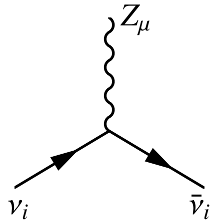


$$= \mp ie\xi M_W$$

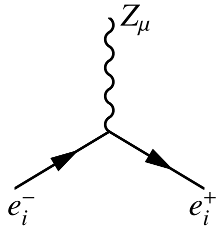


$$= -\frac{i\xi M_Z^2}{v}$$

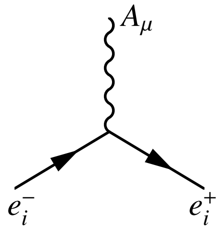
B.4.6 Fermion-Gauge Interactions



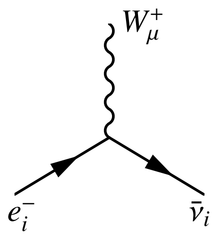
$$= \frac{ig}{4c_W}\gamma^\mu(1-\gamma^5)$$



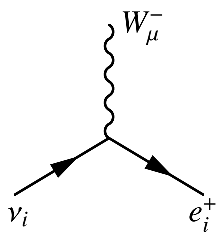
$$= \frac{ig}{4}\gamma^\mu(c_W(3t_W^2-1)-\gamma^5/c_W)$$



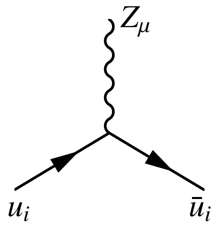
$$= -ie\gamma^\mu$$



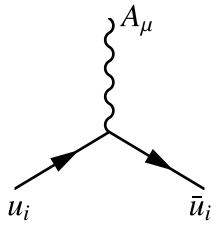
$$= \frac{ig}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$



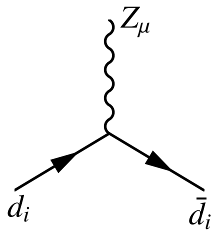
$$= \frac{ig}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$



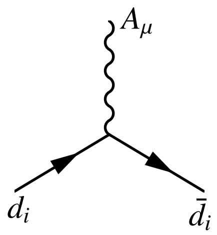
$$= \frac{ig}{4}\gamma^\mu(c_W(1 - 5t_W^2/3) - \gamma^5/c_W)$$



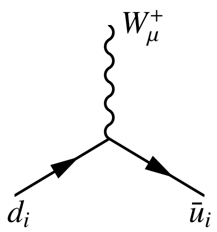
$$= \frac{2ie\gamma^\mu}{3}$$



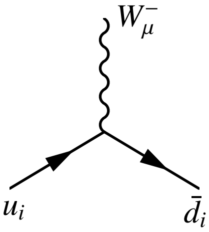
$$= \frac{ig}{4}\gamma^\mu(c_W(t_W^2/3 - 1) + \gamma^5/c_W)$$



$$= -\frac{ie\gamma^\mu}{3}$$



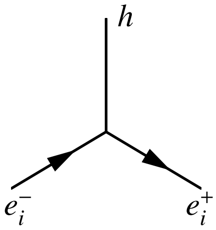
$$= \frac{ig}{2\sqrt{2}}\gamma^\mu(1 - \gamma^5)$$



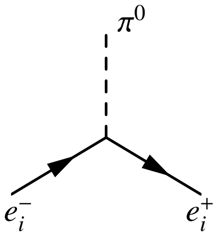
$$= \frac{ig}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

B.4.7 Fermion-Higgs Interactions

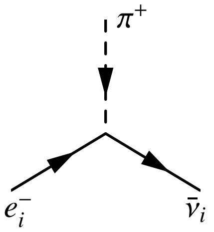
The following diagrams are presented in the MASS EIGENBASIS!!



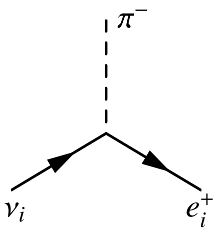
$$= -\frac{im_{e_i}}{v}$$



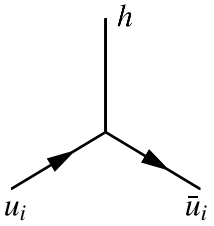
$$= \frac{m_{e_i}}{v}\gamma^5$$



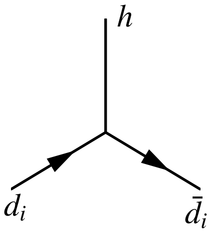
$$= -\frac{iy_i^l}{2}(1+\gamma^5)$$



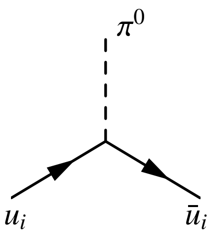
$$= -\frac{iy_i^l}{2}(1-\gamma^5)$$



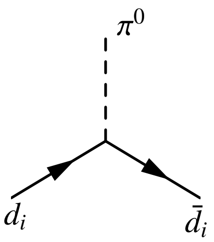
$$= -\frac{im_{u_i}}{v}$$



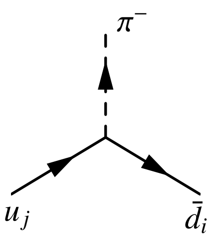
$$= -\frac{im_{d_i}}{v}$$



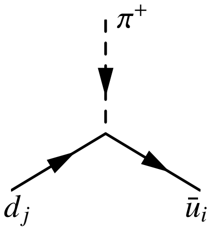
$$= -\frac{m_{u_i}}{v}\gamma^5$$



$$= -\frac{m_{d_i}}{v}\gamma^5$$



$$= \frac{i}{2}(V^\dagger_{\text{CKM}})_{ij}y^u_j(1+\gamma^5) - \frac{i}{2}(V^\dagger_{\text{CKM}})_{ij}y^d_i(1-\gamma^5)$$



$$= -\frac{i}{2}(V_{\text{CKM}})_{ij} y_j^d (1 + \gamma^5) + \frac{i}{2}(V_{\text{CKM}}^\dagger)_{ij} y_i^u (1 - \gamma^5)$$