A First Glimpse of

Quantum Field Theory

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Outline

- Patterns of Physics: Classical / Quantum
- Quantum Field Theory

  Harmonic oscillators / Field theory / Path Integral / Interactions / Summary

- Symmetry and Its Breakdown

  Symmetries / Broken symmetries / Scale symmetry
  Renormalization (semi-)group / c theorem*

WARNING IN ADVANCE

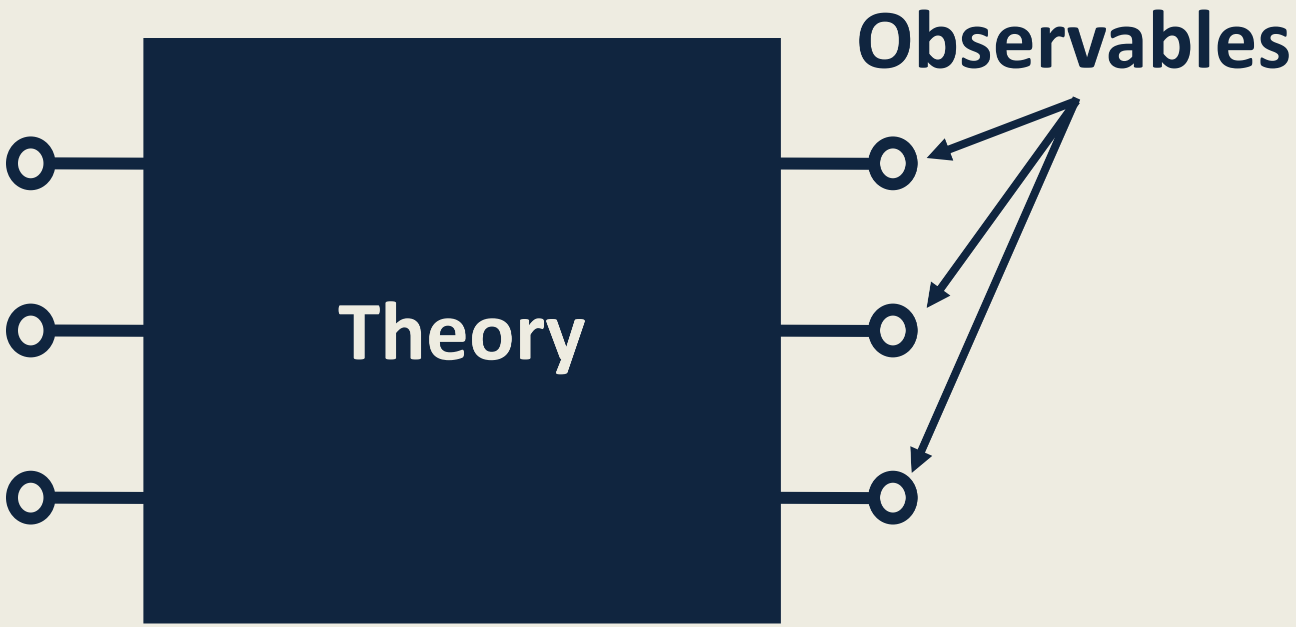
– I myself am not a mathematician!

- **What is a physical theory?**
  - A personal viewpoint.

A black-box problem

Patterns of Physics

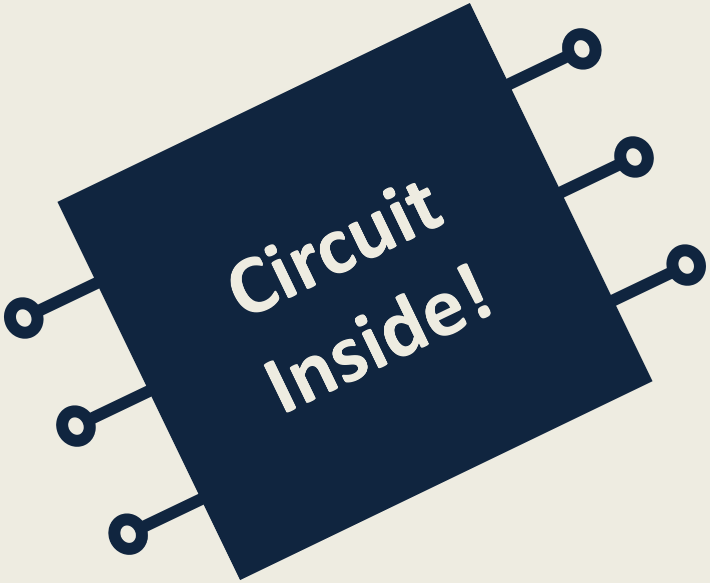
- **What is a physical theory?**
  – A personal viewpoint.



A black-box problem

Patterns of Physics

- **Different patterns / paradigms**



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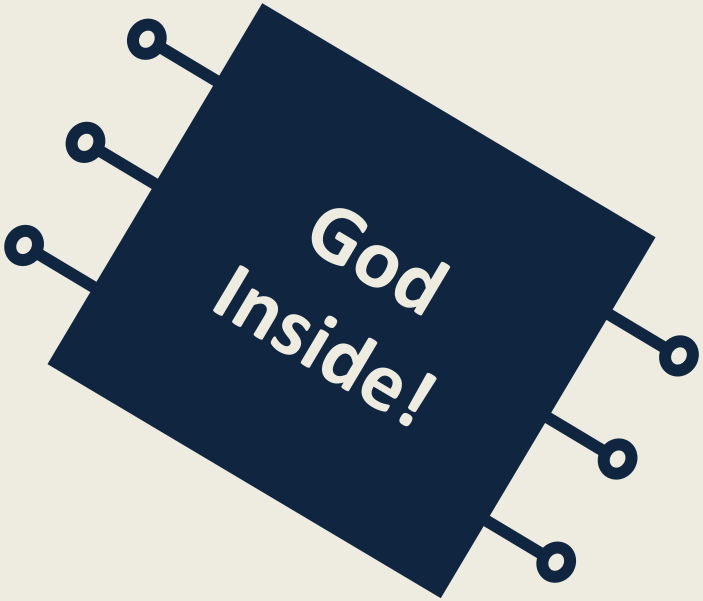


Patterns of Physics

- **Classical Mechanics**
  - Phase space / Canonical variables
  - Hamilton’s equations
  - Observables / Poisson brackets

- **Quantum Mechanics**
  - Hilbert space / Quantum states
  - Schrödinger’s equation
  - Self-adjoint operators / Commutators

Patterns of Physics

- **Quantization: Classical → Quantum**
  - **Key ideas**

    1) To turn observables from functions over the phase space to self-adjoint operators on the Hilbert space.
    2) To turn Poisson brackets into commutators.
  - Loosely speaking, the procedure of (canonical) quantization is a map:

$$\widehat{\ }: f \mapsto \widehat{f},$$

satisfying:

$$[\widehat{f}, \widehat{g}] = \mathrm{i}\widehat{\{f, g\}}.$$

Patterns of Physics

- **Quantization: Classical → Quantum**
- The **Existence** and **uniqueness** of the quantized theory?
  - The Uniqueness (under certain assumptions not satisfied by field theories):
    Stone-von Neumann theorem
    "Counterexamples"
  - The Existence: not guaranteed in general.
    "Counterexamples"

Harmonic Oscillators

- **The classical description**
  - Phase space: $\mathcal{M} = \mathrm{span}\{q, p\}$
  - Hamiltonian: $H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \quad (\omega > 0)$
  - Hamilton's equations:

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\partial H}{\partial p} \\ \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\dfrac{\partial H}{\partial q} \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{q} = p \\ \dot{p} = -\omega^2 q \end{cases}$$

  - Poisson bracket:

$$\{q, p\} = 1, \qquad \{q, q\} = \{p, p\} = 0$$

Harmonic Oscillators

- **The quantum description**
  - Hilbert space: $\mathcal{H} = \mathrm{span}\{|n\rangle\}_{n=0}^{\infty}$
  - Hamiltonian:

$$H = \big(a^\dagger a + \tfrac{1}{2}\big)\omega, \qquad a = \sqrt{\tfrac{\omega}{2}}\,q + \mathrm{i}\sqrt{\tfrac{1}{2\omega}}\,p$$

  - Schrödinger's equations:

$$H|\Psi\rangle = \mathrm{i}\tfrac{\partial}{\partial t}|\Psi\rangle \quad \Rightarrow \quad H|n\rangle = \big(n + \tfrac{1}{2}\big)\omega|n\rangle$$

  - Commutators:

$$[q,p] = \mathrm{i}, \qquad [q,q] = [p,p] = 0$$

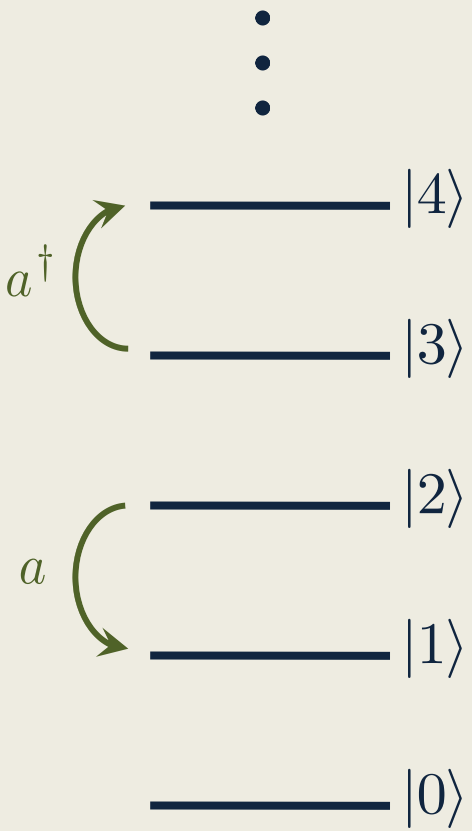
$$\Rightarrow \quad [a,a^\dagger] = 1, \quad [a,a] = [a^\dagger,a^\dagger] = 0$$

Harmonic Oscillators

- **The quantum description**
  - Energy spectrum
  - Vacuum state $|0\rangle$
  - Raising / lowering operators

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a|n\rangle = \sqrt{n}|n-1\rangle \quad (n>0)$$
$$a|0\rangle = 0$$

$$\Rightarrow \quad |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$



Harmonic Oscillators

- **N decoupled harmonic oscillators**

  Hamiltonian $H = \sum_{i=1}^{N} \left(a_i^\dagger a_i + \frac{1}{2}\right)\hbar\omega_i \quad (\omega_i > 0)$

  Commutator $[a_i, a_j^\dagger] = \delta_{ij}$

  – Construction of Hilbert space

    Tensor product construction:

$$\mathcal{H}_\mathrm{T} = \bigotimes_{i=1}^{N} \mathcal{H}_i, \qquad \mathcal{H}_i = \mathrm{span}\{|n_i\rangle\}_{n=0}^{\infty}$$

    Fock space construction:

$$\mathcal{H}_\mathrm{F} = \bigoplus_{k=1}^{\infty} \Big(\otimes_S^k \mathcal{P} \Big), \qquad \mathcal{P} = \mathrm{span}\{q_i = e^{-\mathrm{i}\omega_i t}\}_{i=1}^{N}$$

Harmonic Oscillators

- **N decoupled harmonic oscillators**
  - Unitary equivalence $\mathcal{H}_T \cong \mathcal{H}_F$

    i.e., $\exists U : \mathcal{H}_T \to \mathcal{H}_F$ being unitary
  - An explicit example with $N = 2$

$$\mathcal{H}_T = \mathrm{span}\{|n_1 n_2\rangle\,;\, n_1, n_2 \in \mathbb{N}\}$$

$$\mathcal{H}_F = \mathrm{span}\{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0\rangle\,;\, n_1, n_2 \in \mathbb{N}\}$$

$$U\,:\,|n_1 n_2\rangle \mapsto \frac{1}{\sqrt{n_1! n_2!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0\rangle$$

  - Different ways of counting states.
  - The need for symmetrization.

Field Theory

- **What is a field?**
  - A real-valued field $f \in C^\infty(\mathbb{R}^4); \quad f(t,\mathbf{x}) \in \mathbb{R}$

    E.g., density distribution $\rho(t,\mathbf{x})$
  - Vector-, Lie algebra-, or coset space-valued, etc.

    E.g., Electromagnetic field $\mathbf{E}(t,\mathbf{x}), \quad \mathbf{B}(t,\mathbf{x})$
- **What is a field theory?**
  - Fields as canonical variables.

    E.g., $\mathcal{M} = \mathrm{span}\{\phi_0(\mathbf{x}), \pi_0(\mathbf{x})\}, \quad \phi_0, \pi_0 \in C_0^\infty(\mathbb{R}^3)$
  - A theory of infinitely many degrees of freedom.

Field Theory

- **Free field theory**
  - Linear phase space (no curvature);
  - Quadratic Hamiltonian functional.
- **Classical field theory**

Phase space $\mathcal{M} = \text{span}\{\phi_0(\mathbf{x}), \pi_0(\mathbf{x})\}$

$\phi_0 \equiv \phi|_{t=0}, \quad \pi_0 \equiv \pi|_{t=0}, \quad \phi_0, \pi_0 \in C_0^\infty(\mathbb{R}^3)$

Hamiltonian $H = \int \mathrm{d}^3x \, \frac{1}{2}\big(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\big) \quad (m > 0)$

Hamilton's equations $\dot{\phi} = \pi, \quad \dot{\pi} = (\nabla^2 - m^2)\phi$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0}$$

— Klein-Gordon equation

Field Theory

- **Classical field theory**

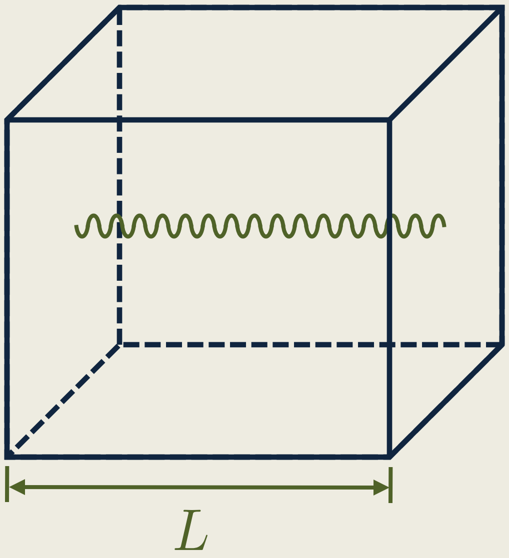
  Solving Klein-Gordon equation in a box with periodic boundary condition (3-torus).

$$\phi(t,\mathbf{x}) = L^{-3/2}\sum_{\mathbf{k}}\phi_{\mathbf{k}}(t)e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \qquad \mathbf{k}\in\{2\pi L^{-1}(n_x,n_y,n_z)\,;\,n_x,n_y,n_z\in\mathbb{Z}\}$$

  The Hamiltonian functional:

$$H = \sum_{\mathbf{k}}\big(|\pi_{\mathbf{k}}|^2 + \omega_{\mathbf{k}}^2|\phi_{\mathbf{k}}|^2\big) \quad (\,\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}\,)$$

  — Free fields are nothing but an infinite number of harmonic oscillators.



Field Theory

- **Quantum field theory**
  - The construction of quantum theory for free fields is fully in parallel with that for harmonic oscillators.

Raising and lowering operators $\phi_{\mathbf{k}} = (2\omega_{\mathbf{k}})^{-1/2}(a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}})$

Commutators $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}$

Vacuum state $|0\rangle \in \mathcal{H}_0; \qquad a_{\mathbf{k}}|0\rangle = 0, \quad \forall \mathbf{k}$

N-particle state $|\mathbf{k}_1 \cdots \mathbf{k}_N\rangle = a^\dagger_{\mathbf{k}_1} \cdots a^\dagger_{\mathbf{k}_N}|0\rangle \in \mathcal{H}_n$

The Hilbert space is again given by Fock space construction

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

So far we have establish the quantum Hilbert space for the free field theory.

BUT THIS IS NOT THE END OF THE STORY !

Path Integral Formulation

- The **prediction** of a quantum theory consists of expectation values of observables (self-adjoint operators).
  - In our case, this can be fully represented by all **n-point Green functions** $(n \in \mathbb{Z}_+)$,

$$G(x_1, \cdots, x_n) = \langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)|0\rangle,$$

  - or equivalently, by the **generating functional** of Green functions (also known as **partition function**),

$$Z[J] = \langle 0|\mathrm{T}\exp\left(\mathrm{i}\int \mathrm{d}^4x\, J(x)\phi(x)\right)|0\rangle.$$

Path Integral Formulation

- We Fourier-transform the partition function $Z[J]$ as

$$Z[J] = \int [\mathrm{d}\varphi]\, e^{\mathrm{i}S[\varphi]} e^{\mathrm{i}\int \mathrm{d}^4x\, J(x)\varphi(x)}.$$

  – Fourier conjugate pairs:

$$\begin{aligned} J &\sim \varphi \\ Z[J] &\sim e^{\mathrm{i}S[\varphi]} \end{aligned}$$

  – This Fourier transformation only has a formal meaning, unless one can define the functional-integral measure $[\mathrm{d}\varphi]$ properly. **The devil is here**.
  – Physicists tend to delay the definition of this measure until it causes troubles. They give the process of "Giving $[\mathrm{d}\varphi]$ a definition" a weird name: **Regularization**.

Path Integral Formulation

Remarks

- The functional $S[\varphi]$ appeared in the Fourier transformed partition function is conventionally called the **action** of the theory.

- In simple cases (e.g., free scalar theory), it can be shown that the action obtained in this way coincides with the one defined in classical Langrange mechanics!

Path Integral Formulation

- Recall that in classical mechanics, the action is an integral of the Lagrangian functional, which in turn can be obtained from Hamiltonian through Legendre transformation:

$$H = \int \mathrm{d}^3x\,\mathcal{H}; \quad \mathcal{H} = \tfrac{1}{2}\big(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\big)$$

$$\Rightarrow \quad \mathcal{L} = \pi\dot{\phi} - \mathcal{H} = \tfrac{1}{2}\big(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\big)$$

$$\Rightarrow \quad S[\phi] = \int \mathrm{d}^4x\,\mathcal{L} = \int \mathrm{d}^4x\,\tfrac{1}{2}\big(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\big).$$

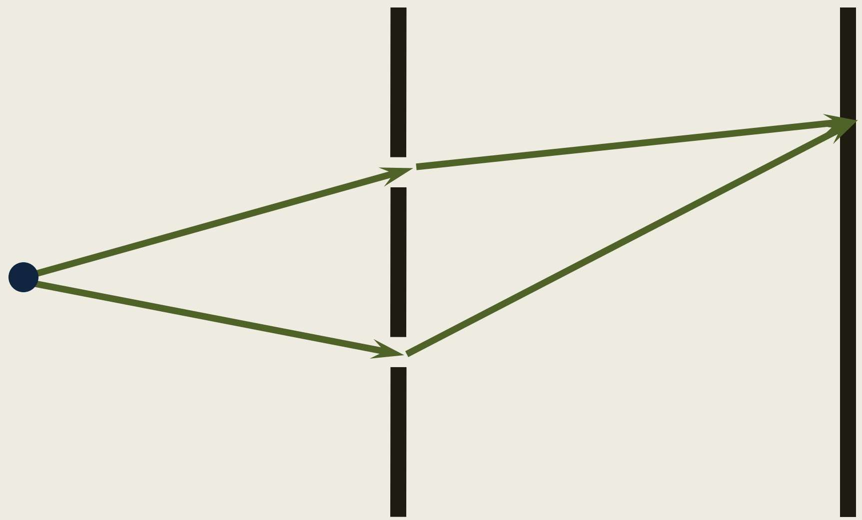
Path Integral Formulation

- This reminds us that one may run the machine backward:
  - We begin with the classical action (rather than Hamiltonian), and use it to define the quantized theory by means of the partition function.
  - All "quantum" information is stored in the path integral measure.

- This is the so-called **functional quantization**, or **path integral quantization**.

Path Integral Formulation

- **Feynman** gives the partition function a beautiful explanation, as "summing over all physical paths". This is the reason for the name "**path integral**".



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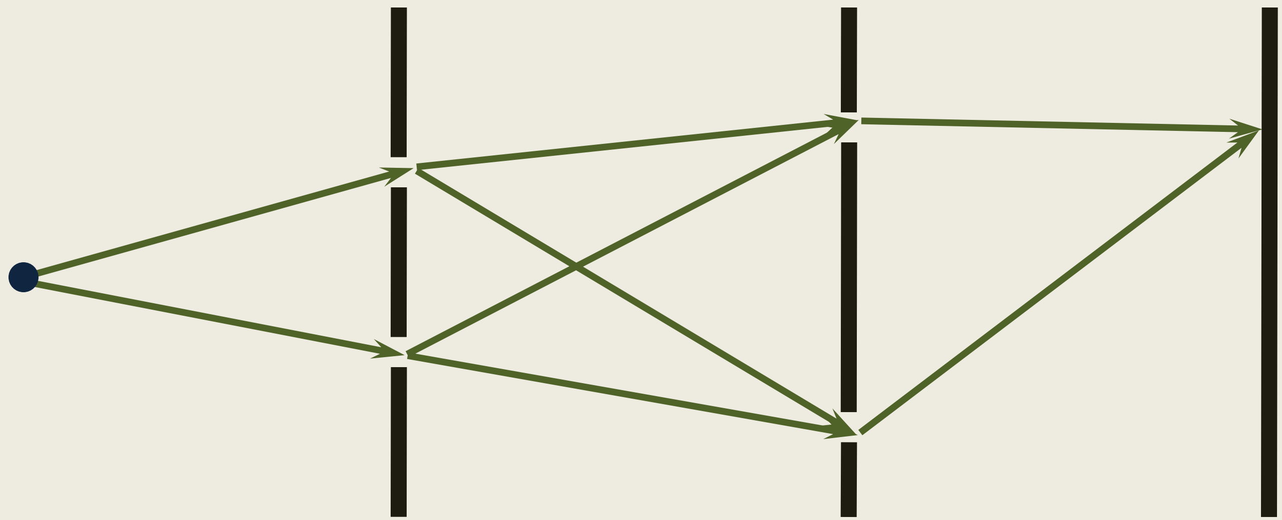


Path Integral Formulation

- **Feynman** gives the partition function a beautiful explanation, as "summing over all physical paths". This is the reason for the name "**path integral**".

Interactions

- Interactions can be conveniently included with the framework of path integral (if we temporarily disregard the definition of the integral measure),

$$S[\phi] = \int \mathrm{d}^4x \, \tfrac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)$$
$$\to \int \mathrm{d}^4x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right).$$

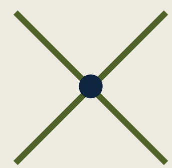
- Then in weakly interacted theories, the observables (Green functions) can be solved perturbatively, and be represented elegantly by **Feynman diagrams**.

Interactions

- **Feynman diagrams**: a simple example

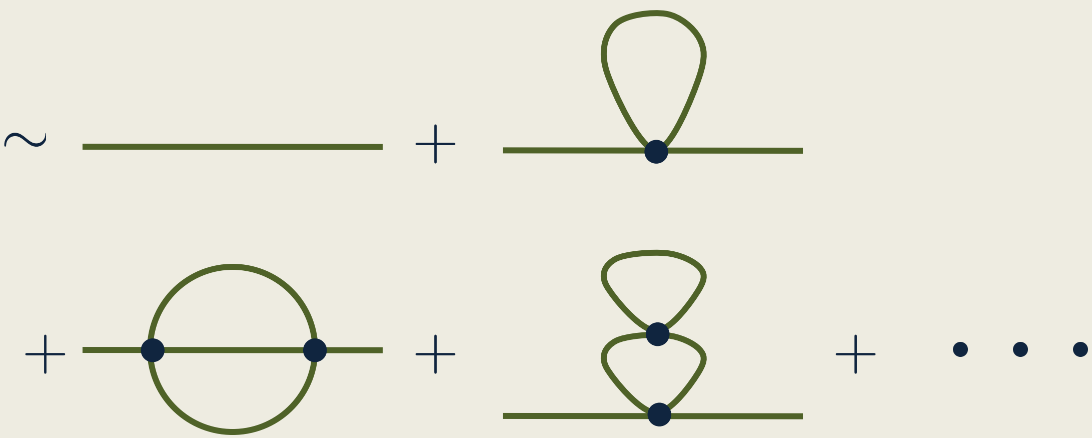
$$S[\phi] = \int \mathrm{d}^4x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\lambda\phi^4 \right)$$

$$= \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon}$$



$$= \mathrm{i}\lambda$$

$$\langle 0|\phi(x)\phi(y)|0\rangle \sim$$

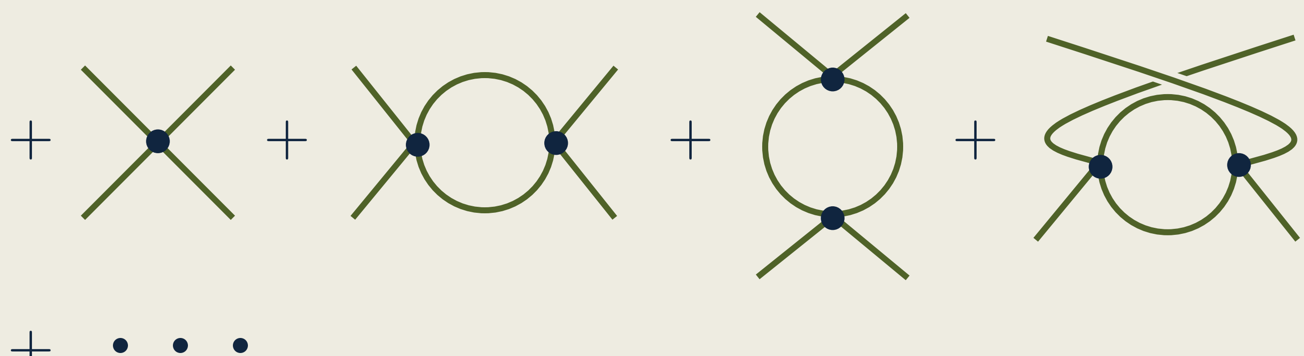


Interactions

- **Feynman diagrams**: a simple example

$$S[\phi] = \int \mathrm{d}^4x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\lambda\phi^4 \right)$$

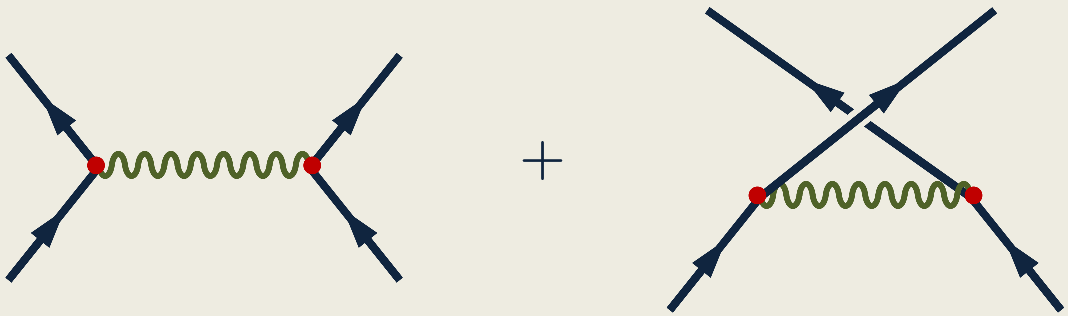
$$\langle 0|\phi(x)\phi(y)\phi(z)\phi(w)|0\rangle \sim$$



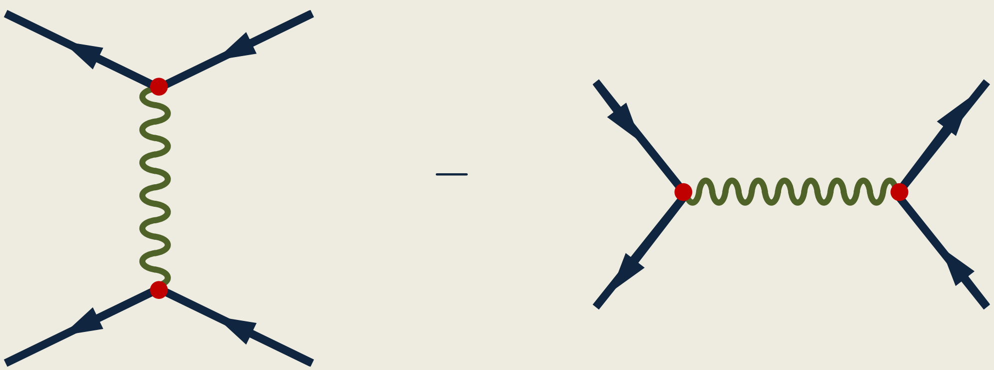
Interactions

- **Feynman diagrams**: more examples

Scattering of two electrons in **Q**(uantum)**E**(lectro)**D**(ynamics)



Scattering of an electron with a positron (anti-electron)



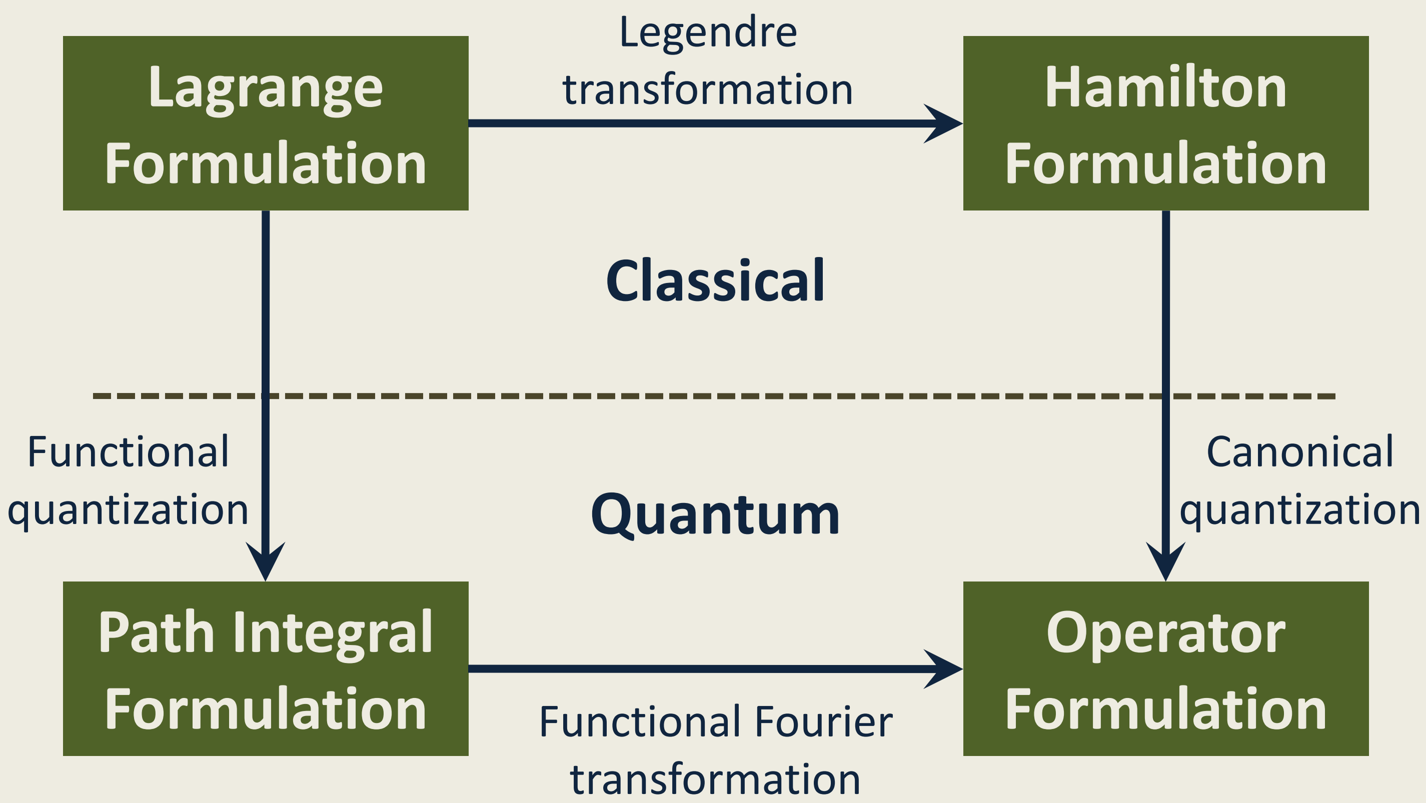
Interactions

Remarks

- Usually, expansion in the number of loops $\sim$ in $\hbar$.
  - Tree diagrams $\sim$ Classical,
  - Loop diagrams $\sim$ Quantum.
- Recall that in the partition function,
  - Action $\sim$ Classical,
  - Integral measure $\sim$ Quantum.
- Integral measure needs regularization, so do loop diagrams, in general.

Summary

- "Commutative diagram" of formulations.



SYMMETRY

… AND ITS BREAKDOWN

Symmetries

- Mathematical structures in QFT:
  - Symmetry,
  - Topology,

  etc.
- Types of symmetries
  - Discrete symmetries / finite groups
  - Continuous symmetries / Lie groups
  - Supersymmetry?

  etc.

Symmetries

- Realization of symmetries

  Classical : Invariance of the **action**.

  Quantum : Invariance of the **partition function**.
- A simple example:

$$S[\phi] = \int \mathrm{d}^4x \, \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi, \qquad Z = \int [\mathrm{d}\phi] \, e^{\mathrm{i}S[\phi]}.$$

$$x \to x + a, \quad a \in \mathbb{R}^4 \qquad \phi(x) \to \phi(x) + \sigma, \quad \sigma \in \mathbb{R}$$

$$x \to \Lambda x, \quad \Lambda \in SO(3,1) \qquad \phi(x) \to \phi(-x)$$

Symmetries

- Examples of continuous symmetries in field theories.

	Spacetime	Internal
Global	Poincaré **Rigid Scale**	Isospin BRST
Local	Diffeomorphism Conformal	Maxwell Yang-Mills

Symmetries

- **Nöther's theorem**
  - "Symmetry implies conservation law."
- More precisely,
  - For each generator of continuous global symmetry, there is a conserved current.
- A "proof".
  - For a global symmetry parameterized by ϵ , the localized transformation of the action must be of the form,

$$\Delta S = \int \mathrm{d}^4 x \, j^\mu(x) \partial_\mu \epsilon(x) = - \int \mathrm{d}^4 x \epsilon(x) \partial_\mu j^\mu(x).$$

  - ΔS must vanish on shell for all $\epsilon(x)$, thus $\partial_\mu j^\mu(x) = 0$.

Broken Symmetries

Three types of broken symmetries

- **Explicitly broken symmetry**
  - Breakdown at the classical level.

- For instance, a mass term in

$$S[\phi] = \int \mathrm{d}^4x \, \tfrac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$$

  breaks the symmetry $\phi(x) \to \phi(x) + a$ explicitly.

- Slightly broken continuous global symmetry implies slightly broken conservation law.

Broken Symmetries

- **Spontaneously broken symmetry**
  - A somewhat misleading name.
  - The symmetry is never broken, but is hidden due to the degenerate vacua.
- Nambu-Goldstone theorem
  - “Spontaneously broken symmetry generate massless particles (Nambu-Goldstone boson).”
  - More precisely, the symmetry is required to be **global** and **internal**. Furthermore, the **Lorentz symmetry** should be manifest.
  - The absence of any of these conditions may alter the result.

Broken Symmetries

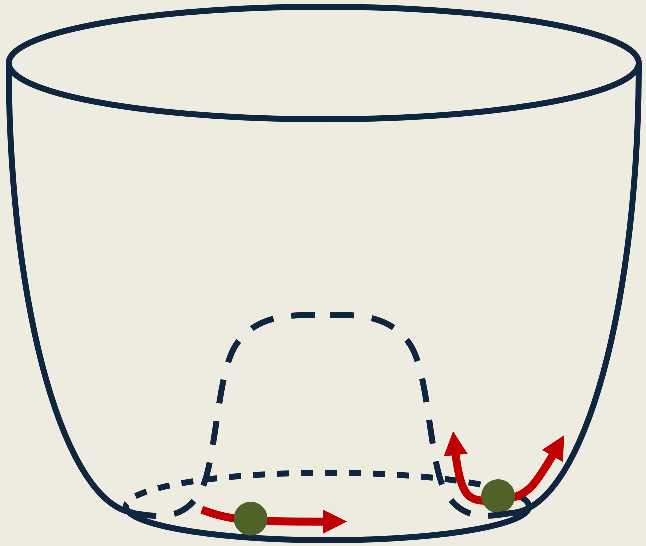
- **Spontaneously broken symmetry**
  - A nearly clichéd example

$$S[\phi] = \int \mathrm{d}^4x \, \tfrac{1}{2}\left(\partial_\mu\phi_i\partial^\mu\phi_i + m^2\phi_i\phi_i - \lambda(\phi_i\phi_i)^2\right) \quad (i = 1, 2)$$

  - The theory contains an infinite number of degenerate vacua:

$$\langle\phi\rangle = \sqrt{m^2/2\lambda}$$

  - Goldstone mode.



Broken Symmetries

- **Anomaly**
  - Another misleading name…
  - For a classical (field) theory with given symmetry, no corresponding quantum theory preserving the symmetry exists.
  - In other words, the symmetry is broken by quantum effects.
  - In terms of path integral, the symmetry is broken by the integral measure.
- We will encounter an example of anomaly when talking about scale symmetry.

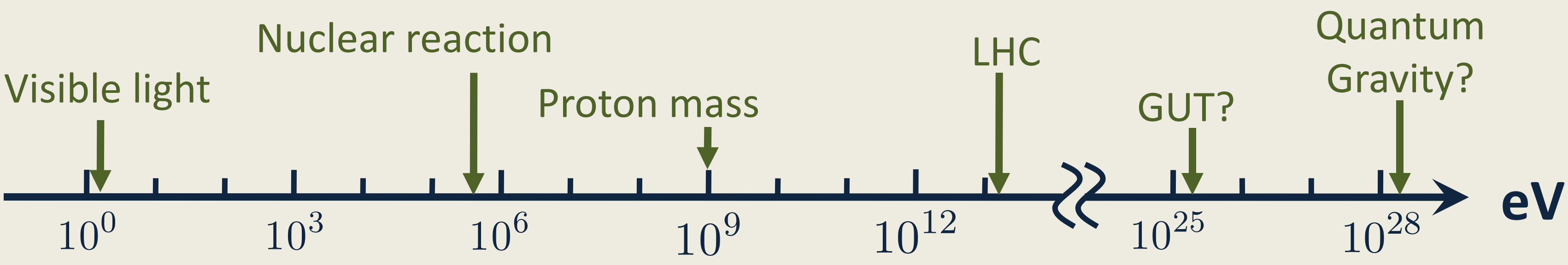
Scale Symmetry

- The world in **natural units**.

$$\boxed{c = \hbar = 1}$$

- There is only a single (independent) unit, which is usually chosen to be the energy.
- Mass dimension [ ] of a quantity.

[energy] = [mass] = [length]$^{-1}$ = [time]$^{-1}$ =…



Scale Symmetry

- The **dimensional analysis** of scalar field theory.

$$S[\phi] = \int \mathrm{d}^4x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \lambda_3\phi^3 - \lambda_4\phi^4 - \cdots \right)$$

$$[S] = 0 \quad \Rightarrow \quad [\mathcal{L}] = 4 \quad \Rightarrow \quad [\phi] = 1$$

$$\Rightarrow \quad [m] = 1, \qquad [\lambda_n] = 4 - n$$

  - Mass parameter has dimension 1, as expected;
  - Cubic-coupling has positive dimension;
  - Quartic-coupling has vanishing dimension;
  - Higher order couplings have negative dimensions.

Scale Symmetry

- This motivates us to define the **scale transformation** as follows:

$$x \to \Omega x; \qquad \phi(x) \to \Omega^{-1}\phi(\Omega x)$$

- Then the action is scale invariant provided that

$$m = 0, \qquad \lambda_n = 0. \quad (n \neq 4)$$

  - In other words, the (classical) scale symmetry is said to be **explicitly broken** by terms other than $\lambda_4\phi^4$.

- Do not confuse scale transformation with dimensional analysis!

Scale Symmetry

- Now we have found a scale invariant classical theory:

$$S[\phi]=\int \mathrm{d}^4x\,\big(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\lambda\phi^4\big).$$

- What if we quantize it?

$$Z[0]=\int[\mathrm{d}\phi]\,e^{\mathrm{i}S[\phi]}.$$

- How to scale the integral measure?
  – We have not defined it yet!
  – The “regularization” is needed.

Scale Symmetry

- It turns out that one can't help but introduce a new scale Λ_{cut} in order to regularize the theory.

- The effect of Λ_{cut} is to exclude the modes with energy roughly higher than this scale, so Λ_{cut} is called the cut-off scale.

- Scale symmetry gets broken then.
  - The breakdown of scale symmetry is said to be a **scale anomaly** or **trace anomaly**.

Renormalization (Semi-)Group

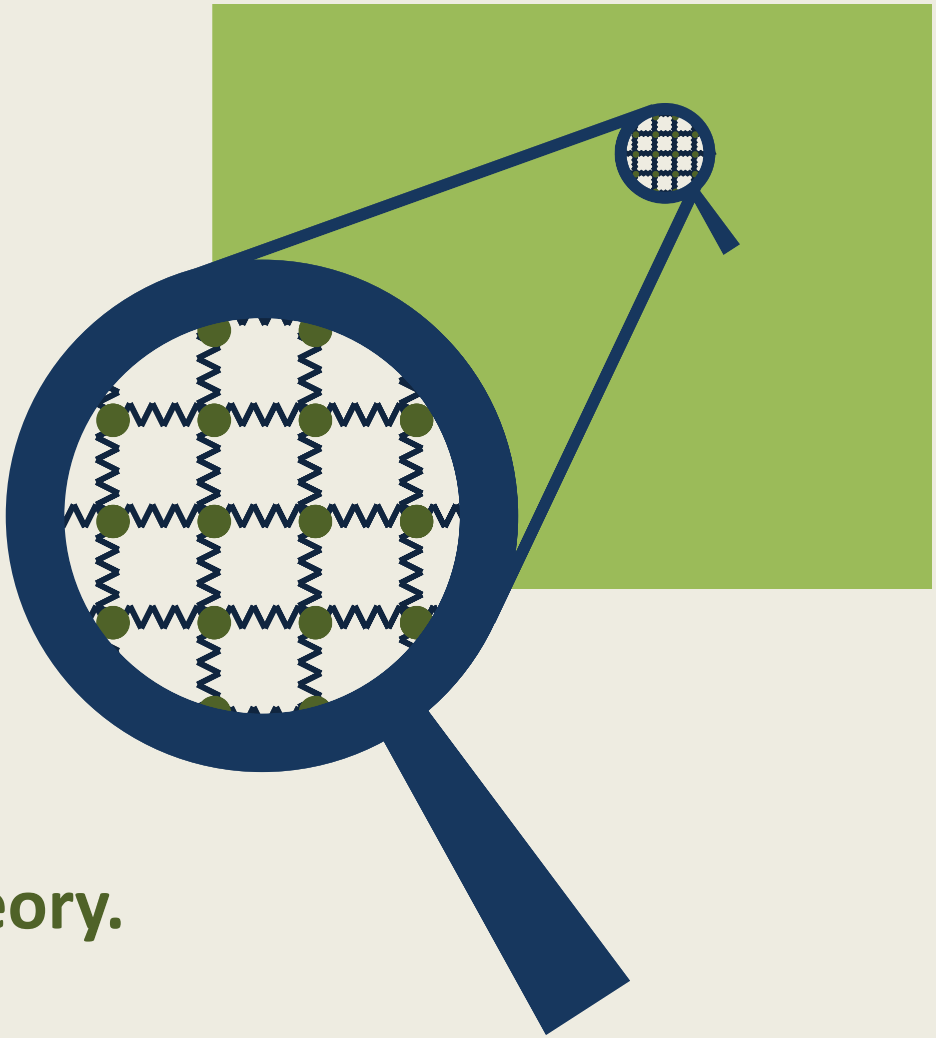
- What if we choose another Λ_{cut} ?
  - For a QFT with Λ_{cut} given, we can obtained a theory with a lower cuf-off scale Λ'_{cut} , by integrating out all modes with energy μ in the layer $\Lambda'_{\text{cut}} < \mu < \Lambda_{\text{cut}}$.
  - The net effect of this manipulation is that parameters in the action get changed.

- A continuous change in cut-off scale yields a flow in the space of theories (parameters), called the **renormalization group (RG) flow**.

Renormalization (Semi-)Group

- Relevant / Irrelevant / Marginal
  - At classical (tree) level in perturbation theory, couplings with (positive , negative , vanishing) dimensions keep (increasing , decreasing , fixed) along the RG flow, the corresponding operators are said to be (relevant , irrelevant , marginal).
  - After turning on quantum effects, marginal operators will in general split into marginally relevant / irrelevant ones.
- Therefore **the scale anomaly manifests itself through the nontrivial RG flow**.

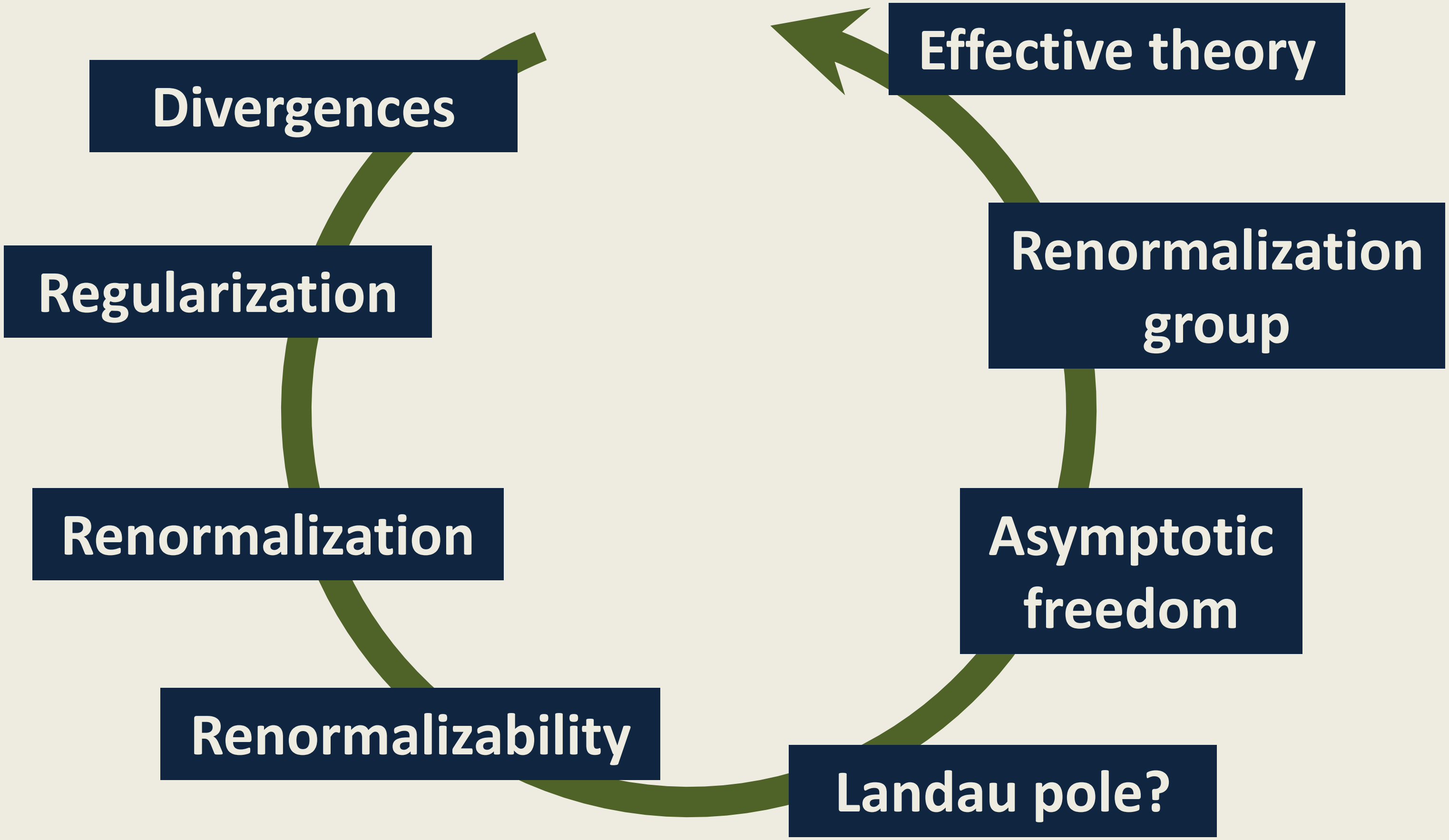
Renormalization (Semi-)Group



- **QFT is an effective theory.**

Renormalization (Semi-)Group

- A Historical Note



Summary

Two lessons

- We had better not treat QFT as a **fundamental** description of nature.
  - What does "fundamental" mean?

- We had better not expect QFT to be an **absolutely precise** description of "visible world".
  - No measurement can be made absolutely precise.

- (In my viewpoint) It makes little sense to talk about ultimate theory or absolute precision in physics.

Summary

One conclusion

- All physical theories are nothing but effective theories!

Thanks for your attention

References

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  - F. Wilczek, Rev. Mod. Phys. 71, S85.
- Introductory textbooks
  - A. Zee.
  - M. E. Peskin & D. V. Schroeder.
- Advanced textbooks
  - S. Weinberg, 3 volumes.

BACK-UP

Recent Developments on RG Flow

- What general properties do RG flows have?
  - In particular, **is RG flow reversible?**

- (A. B. Zomolodchikov, 1986) In 2D, RG flow is a potential flow, and is irreversible. There exists a function (c) of (energy) scale monotonically decreasing along the RG flow.
- (J. L. Cardy, 1988) Is there a c theorem in 4D?
- (Z. Komargodski & A. Schwimmer, 2011) 4D c theorem proved.
- (M. A. Luty *et al.*, 2012) All 4D RG flows approach IR CFTs in perturbation theory.

Recent Developments on RG Flow

- The basic idea of KS proof for 4D c theorem.
  - Put the theory into a conformally flat spacetime.
  - The effective theory relevant for dilaton scattering is fully governed by Weyl anomaly.
  - In particular, the 2 to 2 scattering amplitude of dilatons is proportional to $a_{\mathrm{UV}} - a_{\mathrm{IR}}$.
  - Applying dispersion arguments to show the amplitude is positive definite.